AN
INTRODUCTION
TO
MENSURATION, &c.

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AN

INTRODUCTION
TO
MENSURATION,
AND
PRACTICAL GEOMETRY,

WITH NOTES, CONTAINING THE REASON OF EVERY RULE.

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The FIFTH EDITION, corrected and improved.

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PREFACE.

THE ART OF MEASURING, like all other uſeful inventions, appears to have been the offspring of want and neceſſity; and to have had its origin in thoſe remote ages of antiquity, which are far beyond the reach of credible and authentic hiſtory. Egypt, the fruitful mother of almoſt all the liberal ſciences, is imagined likewiſe to have given birth to GEOMETRY or MENSURATION; it being to the inundations of the Nile that we are ſaid to be indebted for this moſt perfect and delightful branch of human knowledge.

After the overflowings of the river had deluged the country, and all artificial boundaries and land-marks were deſtroyed, there could have been no other method of aſcertaining individual property, than by a previous knowledge of its figure and dimenſions. From this circumſtance, it appears highly probable, that Geometry was firſt known and cultivated by the ancient Egyptians; as being the only ſcience which could adminiſter to their wants, and furniſh them with the aſſiſtance they required. The name itſelf ſignifies properly the *art of meaſuring the earth*; which ſerves ſtill further to confirm this opinion; eſpecially as it is well known that many of the ancient mathematicians applied their geometrical knowledge entirely to that purpoſe; and that even the Elements of Euclid, as they now ſtand, are only the theory from whence we obtain the rules and precepts of our preſent more mechanical practice.

But to trace the ſciences to their firſt rude beginnings, is a matter only of learned curioſity, which could afford but little gratification to readers in general. It is of much more conſequence to the riſing generation to be informed that, in their preſent improved ſtate, they are exceedingly uſeful and important. And in this reſpect, the art I have undertaken to elucidate is inferior to none, arithmetic only excepted. Its uſe in moſt of the different branches of the Mathematics is ſo general and extenſive, that it may be juſtly conſidered as the mother and miſtreſs of all the reſt, and the ſource from whence were derived the various properties and principles to which they owe their exiſtence.

As a teſtimony of this ſuperior excellence, I need only mention a few of thoſe who have ſtudied and improved it; in which illuſtrious catalogue we have the names of Euclid, Archimedes, Thales, Anaxagoras, Pythagoras, Plato, Apollonius, Philo and Ptolemy, amongſt the ancients; and Huygens, Wallis, Gregory, Halley, the Bernoullies, Euler, Liebnitz, and Newton, among the moderns; all of whom applied themſelves to particular parts of it, and greatly enlarged and improved the ſubject. To the latter eſpecially we are indebted for many valuable diſcoveries in the higher branches of the art; which have not only enhanced its dignity and importance, but rendered the practical application of it more general and extenſive.

The degree of eſtimation in which the art was held by theſe and other eminent characters, will, in general, it is apprehended, be thought a ſufficient encomium on its merits. But, for the ſake of young people, and thoſe of a confined education, it may not be amiſs to give a few more inſtances of its advantage, and ſhew that its importance in trade and buſineſs is not inferior to its dignity as a ſcience. Artificers of almoſt all denominations are indebted to this invention for the eſtabliſhment of their ſeveral occupations, and the perfection and value of their workmanſhip. Without its aſſiſtance all the great and noble works of Art would have been imperfect and uſeleſs. By this means the architect lays down his plan, and erects his edifice; bridges are built over large rivers; ſhips are conſtructed; and property of all kinds is accurately meaſured, and juſtly eſtimated. In ſhort, moſt of the elegancies and conveniencies of life owe their exiſtence to this art, and will be multiplied in proportion as it is well underſtood, and properly practiſed.

From this view of the ſubject, it is hardly to be accounted for, that, in a commercial nation, like our own, an art of ſuch general application ſhould have been ſo greatly neglected. Mechanics of all kinds, it is well known, are but ill acquainted with its principles; and thoſe who have been the beſt qualified to afford them any aſſiſtance, have thought it beneath their attention. Till within a few years paſt there could not be found a regular treatiſe upon this ſubject in the Engliſh language. Some particular branches, it is true, had been greatly cultivated and improved; but theſe were only to be found in their miſcellaneous ſtate, interſperſed through a number of large volumes, in the poſſeſſion of but few, and in a form and language totally unintelligible to thoſe for whom they were more immediately neceſſary.

Dr. Hutton was the firſt perſon, in this country, who undertook to collect theſe ſcattered fragments, and to treat of the ſubject in a ſcientific, methodical manner. A ſmall treatiſe by Hawney, and ſome others of little note, had indeed been long in the hands of the public; but theſe were extremely defective, both in matter and method; neither the principles nor practice of the art being properly or clearly explained. Before the publication of the treatiſe abovementioned, Mr. Robertſon's may be conſidered as the only book, of any value, that could be conſulted, either by the artizan or mathematician; and had he given the theory as well as the practice of the art, and diveſted his rules and examples of their algebraical form, there would have been no want of any other elementary treatiſe.

To theſe two writers I am greatly indebted for many things in the following pages, and am ready to acknowledge, that I have uſed an unreſerved freedom in ſelecting from their works, wherever I found them to anſwer my purpoſe. To Dr. Hutton I am particularly obliged, and am ſo far from deſiring to ſupercede the uſe of his performance by this publication, that I only wiſh it to be thought a uſeful introduction to it. His treatiſe is excellent in its kind; and had it been as well calculated for the uſe of the uninformed Artiſt as it is for the Mathematician, the following compendium had certainly never been publiſhed.

The method I have obſerved, in compoſing this work, is that which was uſed in the "*Scholar's Guide to Arithmetic*;" and, as my object has been to facilitate the acquirement of the ſame kind of uſeful knowledge, I am not without hopes of its being received with equal candour and approbation.

In ſchool-books, and thoſe deſigned for the uſe of learners, it has always appeared to me, that plain and conciſe rules, with proper exerciſes, are entirely ſufficient for the purpoſe. In ſcience, as well as in morals, example will ever inforce and illuſtrate precept; for this reaſon an operation, wrought out at length, will be found of more ſervice to beginners than all the tedious directions and obſervations that can poſſibly be given them. From conſtant experience I have been confirmed in this idea; and it is in purſuance of it that I have formed the plan of this publication. I have not been ambitious of adding much new matter to the ſubject; but only to arrange and methodize it in a manner more eaſy and rational than had been done before.

The text part of the work contains the rules in words at length, with examples to exerciſe them;

and, in order that the learner may not be perplexed and interrupted in his progreſs, the remarks and demonſtrations are confined to the notes, and may be conſulted or not, as ſhall be thought neceſſary. To thoſe who would wiſh not to take things upon truſt, but to be acquainted with the grounds and *rationale* of the operations they perform, they will be found extremely ſerviceable; and for this purpoſe I have endeavoured to make them as eaſy as the nature of the ſubject would admit. But they can be conſulted only by ſuch as have made a previous acquaintance with ſeveral other branches of mathematical learning.

Some of the moſt difficult rules relating to the ſurfaces of ſolids, &c. could not be conveniently given, but by means of algebraical theorems; and as this was foreign to my purpoſe, I have not ſcrupled to omit them; being well perſuaded that what is done upon that head will be fully ſufficient to anſwer moſt practical purpoſes. In the Practical Geometry likewiſe, which is prefixed to this treatiſe, ſuch problems only are introduced as were known to be moſt intimately connected with the ſubject. And as this part of the work is a proper aud neceſſary introduction to the reſt, I have ſpared no pains in making it as clear and intelligible as poſſible.

Upon the whole, I have endeavoured to conſult the wants of the learner, more than thoſe of the man of ſcience. And if I have ſucceeded in this reſpect, my purpoſe is anſwered. I have not ſought for reputation as a mathematician, but only to be uſeful as a tutor.

N. B. The favourable reception this Work has met with, has induced me in this Edition to make ſuch Alterations and Additions as have ſince occurred to me, and which are ſuch as I hope will render it ſtill more acceptable to the Public.

Royal Academy, Woolwich,
March 1, 1794.

CON-

CONTENTS.

Page

PRACTICAL GEOMETRY.

Definitions, - - - - - 1
Geometrical problems, - - 9
OF THE MENSURATION OF SUPERFICIES, - 41
To find the area of a parallelogram, - ibid.
To find the area of a triangle, - - 45
To find the ſides of a right-angled triangle, - 48
To find the area of a trapezium, - - 50
To find the area of a regular polygon, - 53
Of a circle, and its parts, - - 57
To find the area of a circular ring, - 80
To find the areas of lunes, - - 82
To find the area of an irregular polygon, - 83
Of the conic ſections, - - 86
Definitions, - - - - - ibid.
To deſcribe an ellipſis, - - - 90
Of an ellipſis, and its parts, - - 91
To deſcribe a parabola, - - 99
Of a parabola, and its parts, - - 101
To conſtruct an hyperbola, - - - 106
Of an hyperbola, and its parts, - - 107

OF THE MENSURATION OF SOLIDS.

Definitions, - - - - 117
To find the ſolidity of a cube, - - 120
To find the ſolidity of a parallelopipedon, - 121
To find the ſolidity of a priſm, - - 122
Of a cylinder, and its parts, - - 124
Of the cone and pyramid, and their parts, - 128
To find the ſolidity of a wedge, - - 139

CONTENTS.

Page
To find the ſolidity of a priſmoid, - - 141
Of a ſphere, and its parts, - - 142
Of a circular ſpindle, and its parts, - - 150
Of a ſpheroid, and its parts, - - 156
Of an elliptic ſpindle, and its parts, - 165
Of a parabolic conoid, and its parts, - 170
Of a parabolic ſpindle, and its parts, - 173
Of an hyperboloid, and its parts, - - 176
Of the regular bodies, - - - 181
Of cylindric rings, - - - 190
Of the weight of balls and ſhells, &c. - 193
Of the piling of ſhot, - - - 208
Of Artificers work, - - - 214
Of vaults and domes, &c. - - 233
Of the Carpenters rule, - - 243
Of board and timber meaſure, - - 247
Of ſpecific gravity, - - - 259
Miſcellaneous queſtions, - - 265
A table of the ſegments of a circle, - - 271

TABLES

TABLES

OF THE

DIFFERENT MEASURES USED IN THIS WORK.

Lineal Meaſures.			*Square Meaſures.*		
12 inches	make	1 foot.	144 inches	make	1 foot.
3 feet	- - -	1 yard.	9 feet	- - -	1 yard.
6 feet	- - -	1 fathom.	36 feet	- - -	1 fathom.
16½ feet, or 5½ yards,	--	1 pole, or rod.	272¼ feet, or 30¼ yds.	--	1 pole, or rod.
40 poles	- -	1 furlong.	1600 poles	- - -	1 furlong.
8 furlongs	- -	1 mile.	64 furlongs	- -	1 mile.

Note, The chain made uſe of in meaſuring land, commonly called Gunter's chain, is 4 poles, or 22 yards in length, and conſiſts of 100 equal links, each link being $\frac{22}{100}$ of a yard, or 7.92 inches long.

An acre of land is alſo equal to 10 ſquare chains; that is, 10 chains in length, and 1 in breadth; or it is 4840 ſquare yards, or 160 ſquare poles, or 100,000 ſquare links.

Note alſo, that in Land Meaſure,			And in Cubic Meaſure,		
40 perches, or ſquare poles	make	1 rood.	1728 inches	make	1 foot.
4 roods	- - - - - - -	1 acre.	27 feet	- - -	1 yard.
			166⅜ yards	- - -	1 pole.

PRAC-

PRACTICAL GEOMETRY.

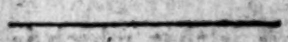
DEFINITIONS.

1. *GEOMETRY* is that ſcience which treats of the deſcriptions and properties of magnitudes in general.

2. A *point* is that which has no parts, or dimenſions.

3. A *line* is length without breadth; and its bounds or extremes are points.

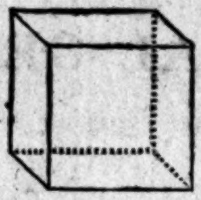
4. A *right* line is that which lies evenly between its extreme points.



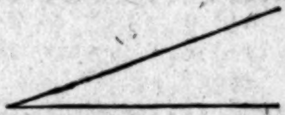
5. A *ſuperficies* is that which has length and breadth only; and its bounds or extremes are lines.

6. A *plane ſuperficies* is that which touches, in every part, any right line that can be drawn in that ſuperficies.

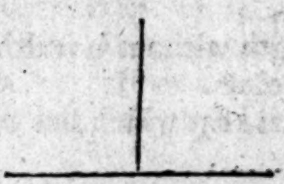
7. A *ſolid* is that which has length, breadth and thickneſs; and its bounds, or extremes, are ſuperficies.



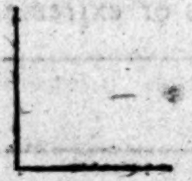
8. A *plane rectilineal angle* is the inclination or opening of two right lines which meet in a point.



9. One line is ſaid to be *perpendicular* to another, when it makes the angles on both ſides of it equal to each other.



10. A *right angle* is that which is formed by two lines that are perpendicular to each other*.



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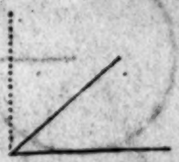
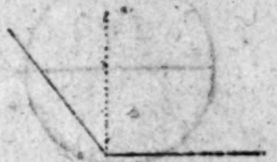
* Any angle differing from a right one, is, by ſome writers, called an *oblique angle*.

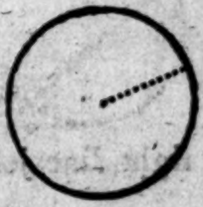
11. An *acute angle* is that which is leſs than a right angle.



12. An *obtuſe angle* is that which is greater than a right angle.



13. A *circle* is a plane figure, formed by the revolution of a right line, about one of its extremities, which remains fixed.*



14. The centre of a circle, is the point about which it is deſcribed; and the circumference is the line or boundary by which it is contained.

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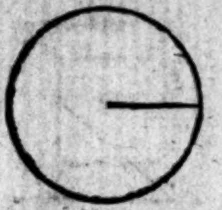
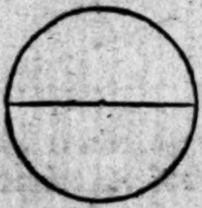
* N. B. The circumference itſelf, for the ſake of conciſeneſs, is ſometimes called a circle.

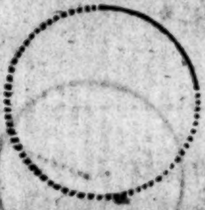
15. The *radius* of a circle is a right line drawn from the centre to the circumference.



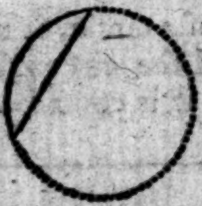
16. The *diameter* of a circle is a right line paſſing through the centre, and terminated both ways by the circumference.



17. An *arc* of a circle is any part of its periphery or circumference.



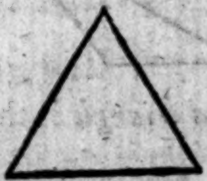
18. A *chord* is a right line joining the extremities of an arc.



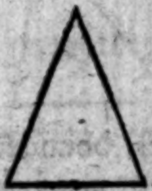
N. B. A ſemicircle is half the circle, and a quadrant the quarter of it.

19. All plane figures bounded by three right lines are called *triangles*.

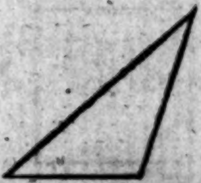
20. An *equilateral triangle* is that whoſe three ſides are all equal.



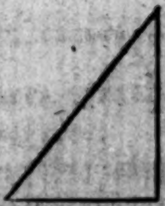
21. An *iſoſceles triangle* is that which has only two of its ſides equal.



22. A *ſcalene triangle* is that which has all its three ſides unequal.



23. A *right-angled triangle* is that which has one right angle.*



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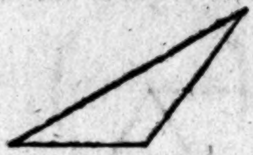
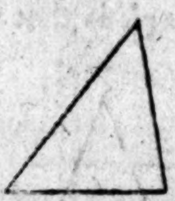
* Any triangle differing from a right angled one, is called *an oblique angled triangle*.

24. An *obtuſe-angled triangle* is that which has one obtuſe angle.



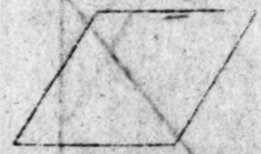
25. An *acute-angled triangle* is that which has all its angles acute.



26. All plane figures, bounded by four right lines, are called *quadrangles*, or *quadrilaterals*.

27. A *ſquare* is a quadrilateral, whoſe ſides are all equal, and its angles all right angles.

28. A *rhombus* is a quadrilateral whoſe ſides are all equal, but its angles not right angles.*



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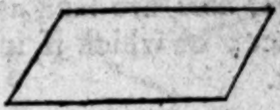
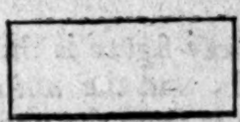
* This figure, by working mechanics, is ſometimes called a lozenge.

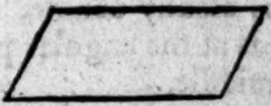
29. A *parallelogram* is a quadrilateral whoſe oppoſite ſides are parallel.



30. A *rectangle* is a parallelogram whoſe angles are all right angles.

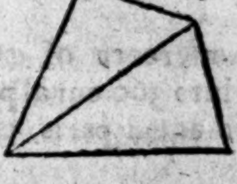


31. A *rhomboid* is a parallelogram whoſe angles are not right angles.



32. All other four-ſided figures, beſides theſe, are called *trapeziums*.

33. A right line joining any two oppoſite angles of a four-ſided figure is called the *diagonal*.



34. All plane figures contained under more than four ſides are called *polygons*.

35. Polygons having five ſides, are called *pentagons*; thoſe of ſix ſides, *hexagons*; thoſe of ſeven, *heptagons*; and ſo on.

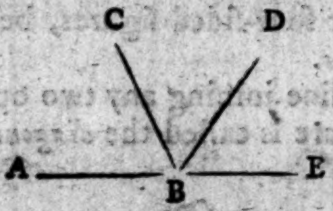
36. A *regular polygon* is that whoſe angles, as well as ſides, are all equal.

37. *Parallel right lines* are ſuch as are every where at an equal diſtance; or which if infinitely produced would never meet.

38. The *baſe* of any figure is that ſide on which it is ſuppoſed to ſtand; and the *altitude* is the perpendicular falling upon it from the oppoſite angle.

39. In a right-angled triangle the ſide oppoſite to the right angle is called the *hypothenuſe*; and the other two ſides are called *legs*.

40. An angle is uſually denoted by three letters, the one which ſtands at the angular point being always to be read in the middle.



*41. The circumference of every circle is ſuppoſed to be divided into 360 equal parts, called *degrees*; each degree into 60 equal parts, called *minutes*; and ſo on.

42. The *meaſure* of any right-lined angle is an arc of a circle contained between the two lines

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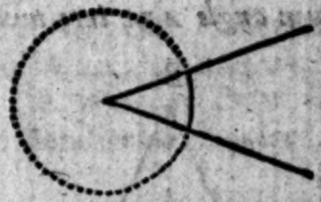
* This and the following definition are uſed only in Practical Geometry.

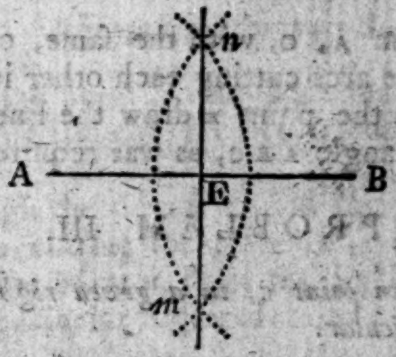
which form that angle, the angular point being the centre.



Note. The angle is eſtimated by the number of degrees contained in the arc; whence a right angle is an angle of 90 degrees, or ¼ of the circumference.

PROBLEM I.*

To divide a given line A B *into two equal parts.*



1. From the points A and B, as centres, with any diſtance greater than half A B, deſcribe arcs cutting each other in *n* and *m*.

2. Through theſe points draw the line *n* E *m*, and the point E, where it cuts A B, will be the middle of the line required.

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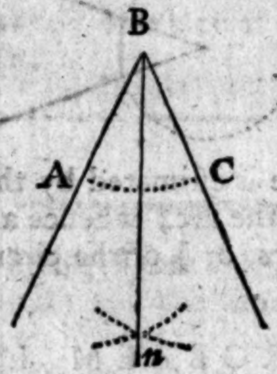
* The demonſtrations of moſt of theſe problems may be found in Euclid's Elements.

PROBLEM II.

To divide a given angle A B C *into two equal parts.*



1. From the point B, with any radius, deſcribe the arc A C.

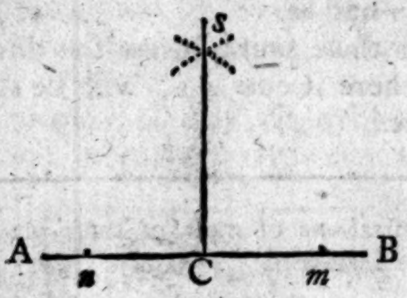
2. And from A, C, with the ſame, or any other radius, deſcribe arcs cutting each other in *n*.

3. Through the point *n* draw the line B *n*, and it will biſect the angle A B C, as was required.

PROBLEM III.

From a given point C, *in a given right line* A B, *to erect a perpendicular.*

CASE I. *When the point is near the middle of the line.*

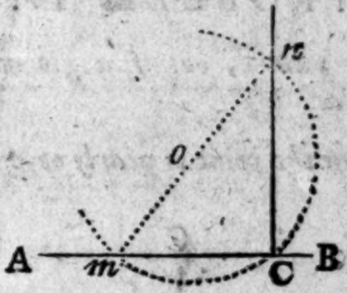


1. On each ſide of the point c take any two equal diſtances c*n*, c*m*.

2. From *n* and *m*, with any radius greater than c*n* or c*m*, deſcribe arcs cutting each other in *s*.

3. Through the point *s*, draw the line *s*c, and it will be the perpendicular required.

Case II. *When the point is at, or near, the end of the line.*

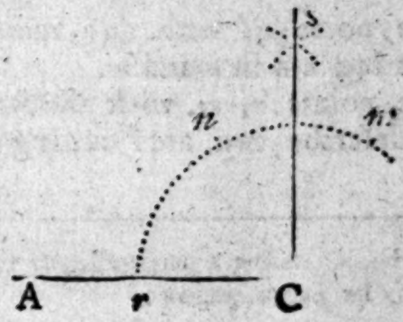


1. Take any point *o*, and with the radius or diſtance *o*c, deſcribe the arc *m*c*n*, cutting AB in *m* and c.

2. Through the centre *o*, and the point *m*, draw the line *mon*, cutting the arc *m*c*n* in *n*.

3. From the point *n*, draw the line *n* c, and it will be the perpendicular required.

Another method.



1. From the point C, with any radius, deſcribe the arc *rnm*, cutting the line A C in *r*.

2. With the ſame radius, and *r* as a centre, croſs the arc in *n*; and from *n*, in like manner, croſs it in *m*.

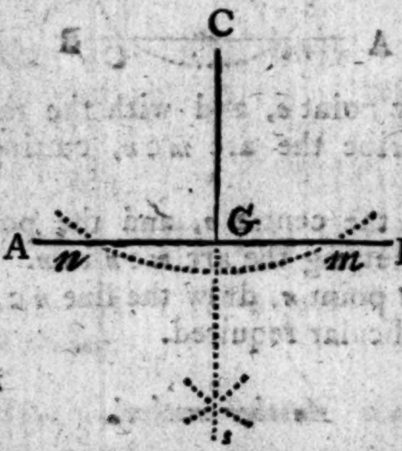
3. From the points *n* and *m*, with the ſame, or any other radius, deſcribe arcs cutting each other in *s*.

4. Through the point *s*, draw the line *s* C, and it will be the perpendicular required.*

PROBLEM IV.

From a given point C, *out of a given line* AB, *to let fall a perpendicular.*

CASE I. *When the point is nearly oppoſite to the middle of the line.*



1. From the point C, with any radius, deſcribe the arc *nm*, cutting A B in *n* and *m*.

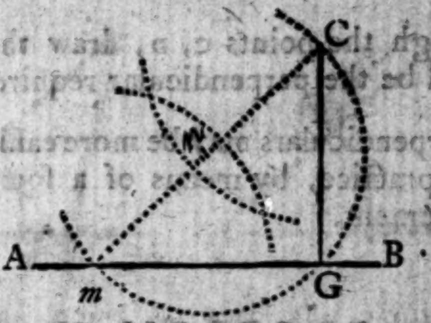
2. From the points *n*, *m*, with the ſame, or any other radius, deſcribe two arcs cutting each other in *s*.

---

* Another method of raiſing a perpendicular from any point in a given line may be ſeen at page 40.

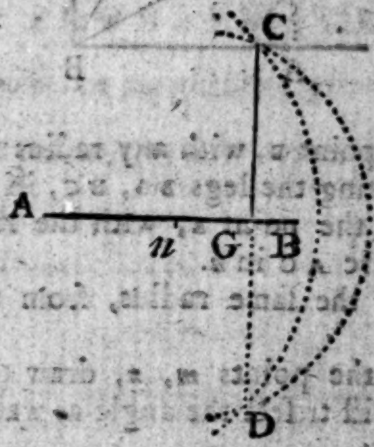
3. Through the points C, *s*, draw the line C G *s*, and C G will be the perpendicular required.

CASE II. *When the point is nearly oppoſite to the end of the line.*



1. To any point *m*, in the line A B, draw the line C *m*.
2. Biſect the line C *m*, or divide it into two equal parts, in the point *n*.
3. From *n*, with the radius *nm*, or *n* C, deſcribe the arc C G *m*, cutting A B in G.
4. Through the point C, draw the line C G, and it will be the perpendicular required.

Another method.



1. From A, or any other point in A B, with the radius A C, deſcribe the arc C D.

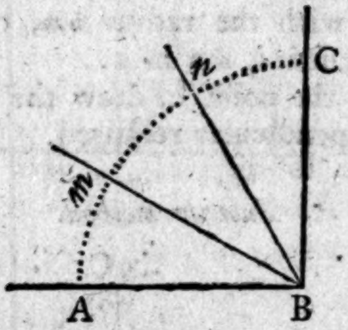
2. And from any other point *n*, in A B, with the radius *n* C, deſcribe another arc cutting the former in C, D.

3. Through the points C, D, draw the line C G D, and C G will be the perpendicular required.

N. B. Perpendiculars may be more eaſily raiſed, and let fall, in practice, by means of a ſquare, or other proper inſtrument.

PROBLEM V.

To triſect, or divide a right angle ABC into three equal parts.



1. From the point B, with any radius B A, deſcribe the arc A C, cutting the legs B A, B C, in A, C.

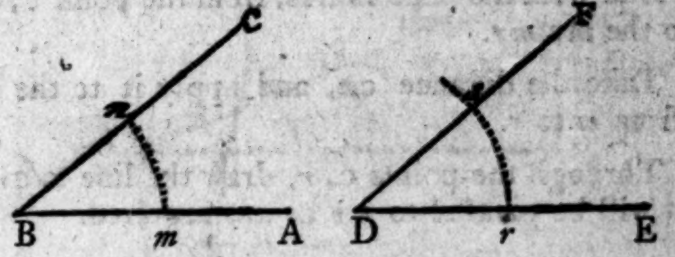
2. And from the point A, with the radius A B or B C, croſs the arc A C in *n*.

3. Alſo with the ſame radius, from the point C, croſs it in *m*.

4. Through the points *m*, *n*, draw the lines B *m*, B *n*, and they will triſect the angle as was required.

PROBLEM VI.

At a given point D *to make an angle equal to a given angle ABC.*



1. From the point B, with any radius, deſcribe the arc *nm*, cutting the legs B A, B C, in the points *m*, *n*.

2. Draw the line D E, and from the point D, with the ſame radius as before, deſcribe the arc *rs*.

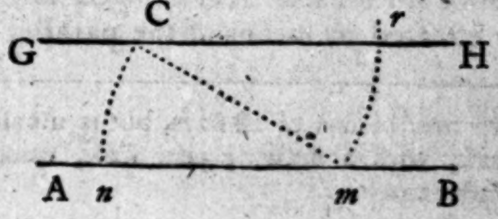
3. Take the diſtance *mn*, on the former arc, and apply it to the arc *rs*, from *r* to *s*.

4. Through the points D *s*, draw the line D F, and the angle E D F will be equal to the angle A B C as was required.

PROBLEM VII.

To draw a line parallel to a given line AB.

Case I. *When the parallel line is to paſs through a given point C.*



1. To A B, from the point C, draw any right line C*m*.

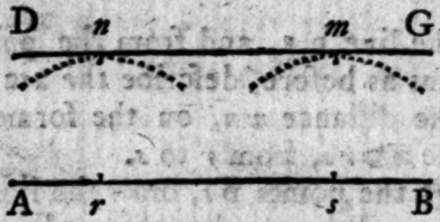
2. From the point *m*, with the radius *m* C, deſcribe the arc C*n*, cutting A B in *n*.

3. And with the ſame radius, from the point C, deſcribe the arc *m r*.

4. Take the diſtance C*n*, and apply it to the arc *mr*, from *m* to *r*.

5. Through the points C, *r*, draw the line G C *r* H, and it will be parallel to A B as was required.

CASE II. *When the parallel line is to be at a given diſtance from AB.*



1. From any two points *r*, *s*, in the line A B, with a radius equal to the given diſtance, deſcribe the arcs *n*, *m*.

2. Draw the line D G, to touch thoſe arcs without cutting them, and it will be parallel to A B as was required.

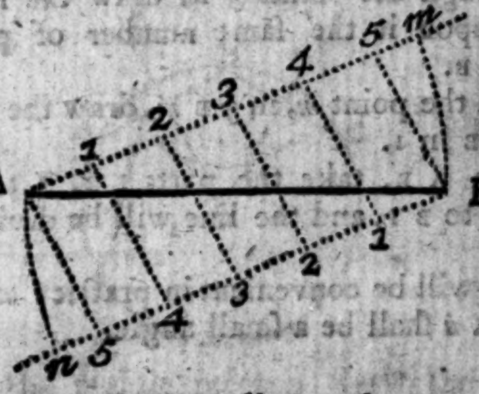
N. B. The former caſe of this problem, as well as ſeveral other operations in Practical Geometry, may be more eaſily effected by means of the parallel ruler.*

---

* This ruler may be had of all ſizes, but is uſually put into a portable caſe, with a drawing-pen, ſcale, compaſſes, and other uſeful inſtruments.

PROBLEM VIII.

To divide a given line A B *into any propoſed number of equal parts.*



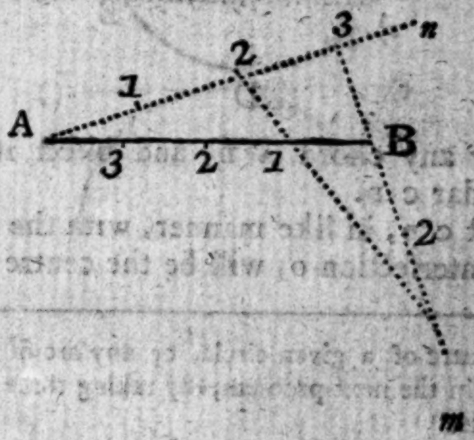
1. From one end of the line A, draw A *m*, making any angle with A B; and from B, the other end, draw B *n*, making an equal angle A B *n*.

2. In each of the lines A *m*, B *n*, beginning at A and B, ſet off as many equal parts, of any length, as A B is to be divided into.

3. Join the points A 5, 1 4, 2 3, &c. and A B will be divided as was required.

Note. B *n* may be drawn parallel to A *m*, by means of a parallel ruler.

Another method.



1. From the point A draw any line A *n*, and ſet on it as many equal parts, wanting one, as A B is to be divided into.

2. Through the points 3 B, draw the line 3 B *m*, and take upon it the ſame number of parts, each equal to 3 B.

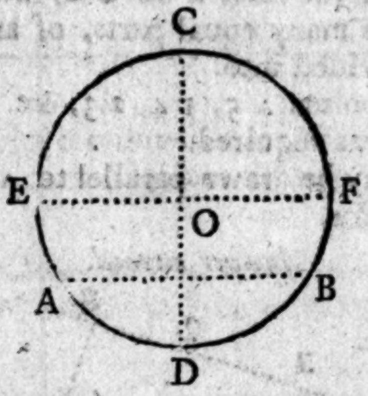
3. From the point *m*, in 3 B *m*, draw the line *m* 1 2, cutting A B in 1.

4. Upon A B, take the parts 1 2, 2 3, and 3 A, each equal to B 1, and the line will be divided as was required.

Note. It will be convenient in practice to draw A *n* ſo that B A *n* ſhall be a ſmall angle.

PROBLEM IX.

*To find the centre of a given circle, or one already deſcribed.**



1. Draw any chord A B, and biſect it with the perpendicular C D.

2. Biſect C D, in like manner, with the chord E F, and their interſection O, will be the centre required.

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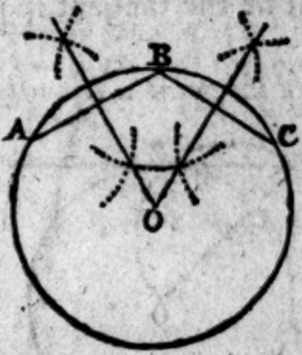
* The centre of a given circle, or any arc of it, may alſo be found as in the next problem, by taking three points in the circumference.

PRO-

PROBLEM X.

To deſcribe the circumference of a circle through three given points, A, B, C.



1. From the middle point B, draw the lines, or chords, B A and B C.

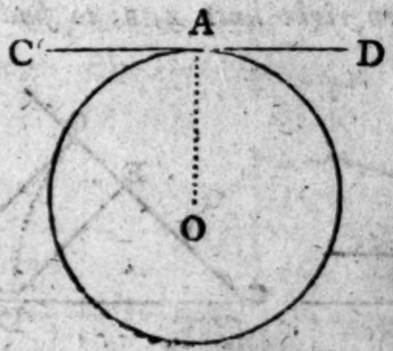
2. Biſect theſe chords perpendicularly, with lines meeting each other in O.

3. From the point of interſection O, with the diſtance O A, O B, or O C, deſcribe the circle A B C, and it will be that required.

PROBLEM XI.

To draw a tangent to a given circle, that ſhall paſs through a given point A.

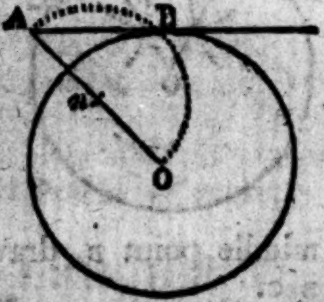
Case I. *When the point* A *is in the circumference of the circle.*



1. From the given point A, to the centre of the circle, draw the radius O A.

2. Through the point A, draw C D perpendicular to O A, and it will be the tangent required.

[illegible] II. *When the point A is without the circle.*



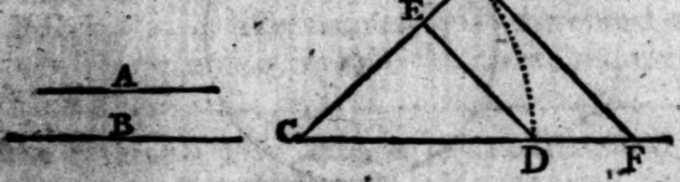
1. To the point A, from the centre O, draw the line O A, and bisect it in *n*.

2. From the point *n*, with the radius *n* A, or *n* O, describe the semi-circle A B O, cutting the given circle in B.

[illegible] Through the points A, B, draw the line A B, [illegible] will be the tangent required.

PROBLEM XII.

To two given right lines A, B, to find a third proportional.



1. From the point C draw two right lines, making any angle FCG.

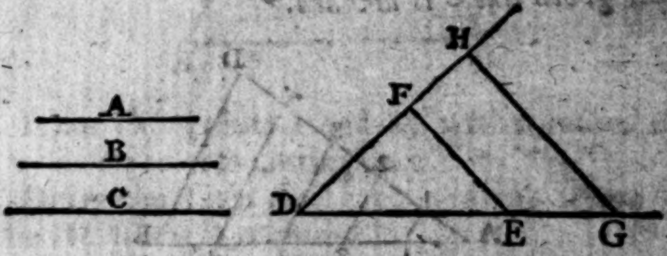
2. In theſe lines take CE equal to the firſt term A, and CG, CD, each equal to the ſecond term B.

3. Join ED, and draw GF parallel to it; and CF will be the third proportional required.

That is CE (A) : CG (B) :: CD (B) : CF.

PROBLEM XIII.

To three given right lines A, B, C, to find a fourth proportional.



1. From the point D draw two right lines, making any angle GDH.

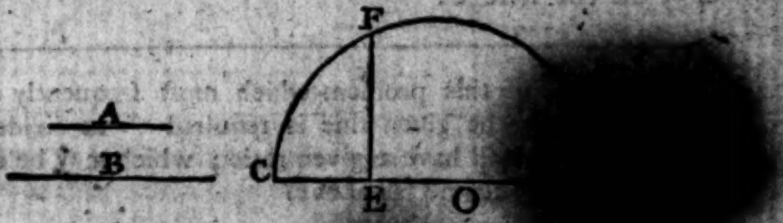
2. In theſe lines take DF equal to the firſt term A, DE equal to the ſecond B, and DH equal to the third C.

3. Join FE, and draw HG parallel to it, and DG will be the fourth proportional required.

That is DF (A) : DE (B) :: DH (C) : DG.

PROBLEM XIV.

Between two given right lines A, B, to find a mean proportional.



1. Draw any right line, in which take CE equal to A, and ED equal to B.

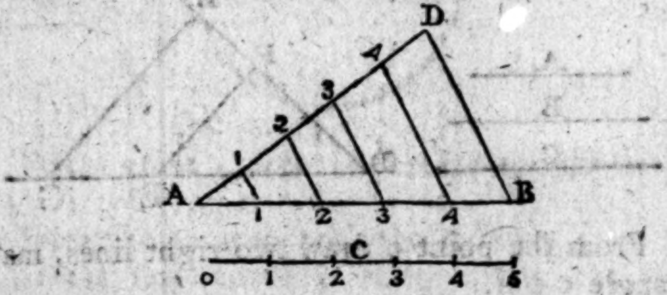
2. Biſect CD, in O, and with OD or OC, as radius, deſcribe the ſemi-circle CFD.

3. From the point E draw EF perpendicular to CD, and it will be the mean proportional required.

That is CE (A) : EF :: EF : ED (B).

PROBLEM XV.

To divide a given line AB in the ſame proportion that another given line C is divided.[*]



1. From the point A draw AD equal to the given line C, and making any angle with AB.

2. To AD apply the ſeveral diviſions of C, and join DB.

3. Draw the lines 44, 33, &c. each parallel to DB, and the line AB will be divided as was required.

That is the parts A1, 12, 23, 34, 4B, on the line AB, will be proportional to the parts 01, 12, 23, 34, 45, on the line C.

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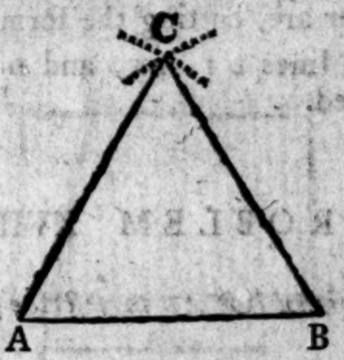
[*] The caſe of this problem which moſt frequently occurs, is that in which the given line is required to be divided into two parts that ſhall have a given ratio; which may be done in nearly the ſame manner as above.

PROBLEM XVI.

Upon a given right line AB, to make an equilateral triangle.



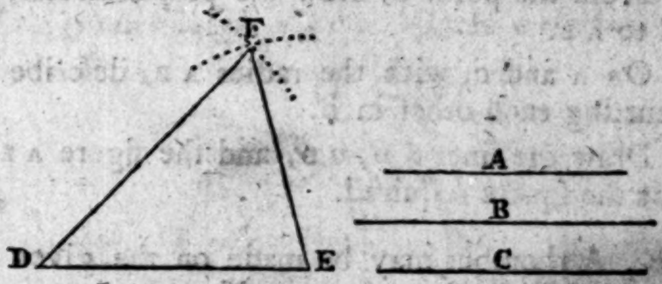
1. From the points A and B, with a radius equal to A B, deſcribe arcs cutting in C.

2. Draw the lines A C, B C, and the figure A C B will be the triangle required.

Note. An iſoſceles triangle may be formed in the ſame manner, by taking any diſtance for radius.

PROBLEM XVII.

To make a triangle whoſe three ſides ſhall be reſpectively equal to three given lines A, B, C.[*]



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[*] The three given lines muſt be each of ſuch a length that any two of them taken together ſhall be greater than the third.

1. Draw a line D E equal to one of the given lines C.

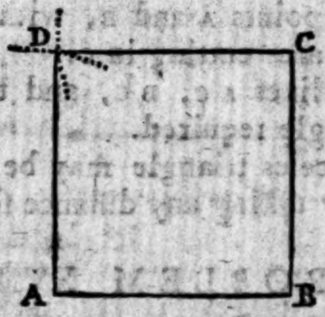
2. On the point D, with a radius equal to B, deſcribe an arc.

3. And on the point E, with a radius equal to A, deſcribe another arc, cutting the former in F.

4. Draw the lines D F, E F, and D F E will be the triangle required.

PROBLEM XVIII.

Upon a given line A B *to deſcribe a ſquare.*



1. From the point B, draw B C perpendicular, and equal to A B.

2. On A and C, with the radius A B, deſcribe two arcs cutting each other in D.

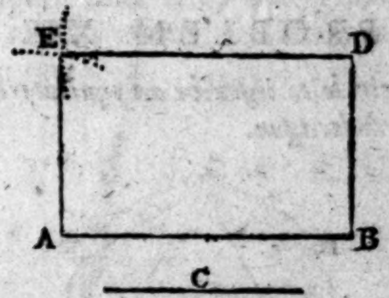
3. Draw the lines A D, C D, and the figure A B C D will be the ſquare required.

Note. A rhombus may be made on the given line A B in exactly the ſame manner, if B C be drawn with the proper obliquity, inſtead of perpendicularly.

PRO-

PROBLEM XIX.

To deſcribe a rectangle, whoſe length and breadth ſhall be equal to two given lines A B *and* C.



1. At the point B, in the given line A B, erect the perpendicular B D, and make it equal to C.

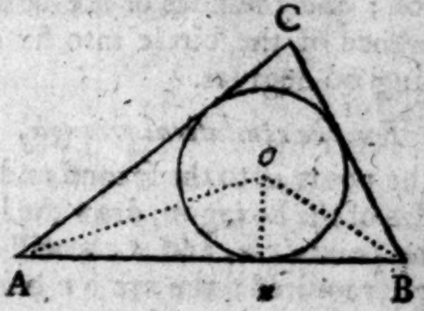
2. From the points D, A, with the radii A B and C, deſcribe two arcs cutting each other in E.

3. Join E A and E D, and A B D E will be the rectangle required.

Note. A parallelogram may be deſcribed in nearly the ſame manner.

PROBLEM XX.

To a given triangle A B C *to inſcribe a circle.*

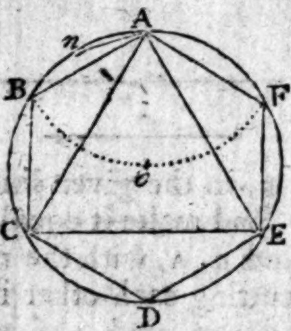


1. Biſect the angles A and B with the lines A O and B O.

2. From the point of interſection O let fall the perpendicular O *n*, and it will be the radius of the circle required.

PROBLEM XXI.

In a given circle to inſcribe an equilateral triangle, an hexagon, or a dodecagon.



For the hexagon.

1. From any point A as a centre, with a diſtance equal to the radius A *o*, deſcribe the arc B *o* F.

2. Join the points A B, or A F, and either of theſe lines being carried ſix times round the circle will form the hexagon required.

That is, the radius of the circle is equal to the ſide of the hexagon; and the ſides of the hexagon divide the circumference of the circle into ſix equal parts, each containing 60 degrees.

For the equilateral triangle.

1. From the point A, to the ſecond and fourth diviſions, or angles of the hexagon, draw the lines A C, A E.

2. Join the points C E, and A C E will be the equilateral triangle required; the arc A C being one third of the circumference, or 120 degrees.

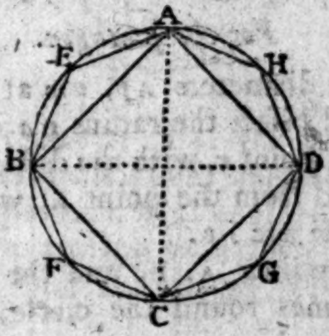
For the dodecagon.

Biſect the arc A B of the hexagon in the point *n*, and the line A *n* being carried twelve times round the circumference, will form the dodecagon required, the arc A *n* being 30 degrees.

If A *n* be again biſected, a polygon may be formed of 24 ſides; and by another biſection a polygon of 48 ſides; and ſo on.

PROBLEM XXII.

To deſcribe a ſquare, or an octagon, in a given circle.



For the ſquare.

1. Draw the diameters B D and A C, interſecting each other at right angles.

2. Join the points A B, B C, C D, and D A, and A B C D will be the ſquare required.

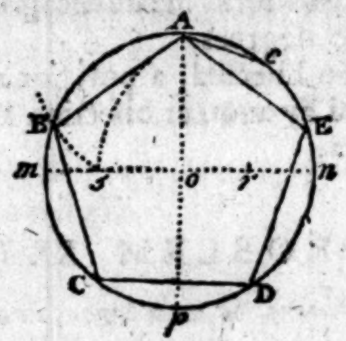
For the octagon.

Biſect the arc A B of the ſquare in the point E, and the line A E being carried eight times round the circumference, will form the octagon.

If the arc A E be again biſected, a polygon may be formed of 16 ſides; and by another biſection a polygon of 32 ſides; and ſo on.

PROBLEM XXIII*.

To inſcribe a pentagon, or decagon, in a given circle.



For the pentagon.

1. Draw the diameters A*p*, *nm* at right angles to each other, and biſect the radius O*n* in *r*.

2. From the point *r*, with the diſtance *r* A, deſcribe the arc A*s*, and from the point A, with the diſtance A*s*, deſcribe the arc *s* B.

3. Join the points A, B, and the line A B being carried five times round the circle, will form the pentagon required.

For the decagon.

Biſect the arc A B of the pentagon in *c*, and the line A*c* being carried ten times round the circumference will form the decagon required.

If the arc A*c* be again biſected, a polygon of 20 ſides may be formed; and by another biſection, a polygon of 40 ſides; and ſo on.

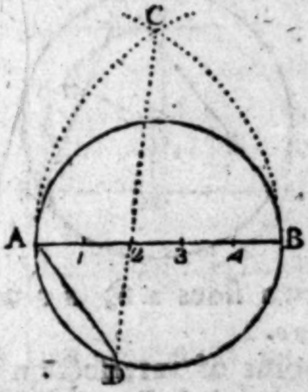
---

* Beſides the figures here conſtructed, and thoſe ariſing from thence by continual biſections, or taking the differences, no other regular polygon can be deſcribed, from any known method *purely geometrical.*

PRO-

PROBLEM XXIV.*

In a given circle to inſcribe any regular polygon.



1. Draw the diameter A B, which divide into as many equal parts as the figure has ſides.

2. From the points A B, as centres, with the radius A B, deſcribe arcs croſſing each other in C.

3. From the point C, through the ſecond diviſion of the diameter, draw the line C D.

4. Join the points A, D, and the line A D will be the ſide of the polygon required.

Note. In this conſtruction A D is the ſide of a pentagon.

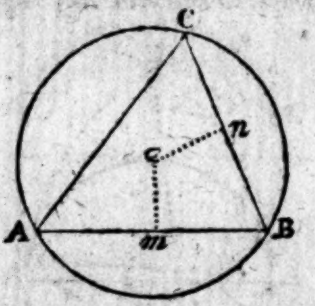
Another method, ſomething more accurate, is by erecting a perpendicular from the centre, of ſuch a length that the part without the circle ſhall be equal to $\frac{3}{4}$ of that within, and drawing a line from its extremity through the ſecond diviſion as before.

---

* This conſtruction is the invention of *Renaldinus*, and was firſt given in his 2d Book *De Reſol. &c. Comp. Mathem.* page 367. The rule for polygons in general is only an approximation, but holds true in the equilateral triangle and hexagon.

PROBLEM XXV.

About a given triangle ABC *to circumſcribe a circle.*



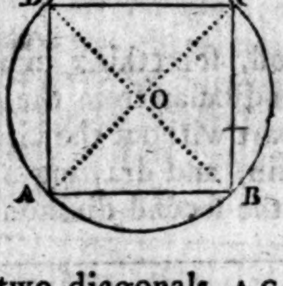
1. Biſect the two ſides AB, BC with the perpendiculars *mo* and *no*.

2. From the point of interſection *o*, with the diſtance OA, OB or OC, deſcribe the circle ACB, and it will be that required.

If any two of the angles be biſected, inſtead of the ſides, the interſection of the lines will alſo give the centre of the circle.

PROBLEM XXVI.

About a given ſquare ABCD *to circumſcribe a circle.*

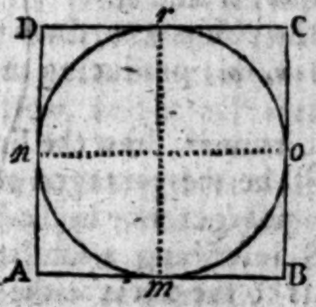


1. Draw the two diagonals AC and BD, interſecting each other in O.

2. From the point O, with the diſtance OA, OB, OC, or OD, deſcribe the circle ABCD, and it will be that required.

PROBLEM XXVII.

To circumſcribe a ſquare about a given circle.

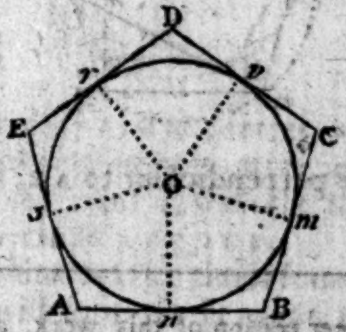


1. Draw any two diameters *no* and *rm* at right angles to each other.

2. Through the points *m o r n*, draw the lines AB, BC, CD, and DA, perpendicular to *rm* and *no*, and ABCD will be the ſquare required.*

PROBLEM XXVIII.

About a given circle to circumſcribe a pentagon.



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* If each of the quadrants *rn*, *nm*, *mo* and *or* be biſected, and tangents be drawn to thoſe points, the circumſcribing figure will be an octagon.

1. Inſcribe a pentagon in the circle; or, which is the ſame thing, find the points *n, m, v, r, s*, as in Problem XXIII.

2. From the centre *o*, to each of theſe points, draw the radii *on, om, ov, or* and *os*.

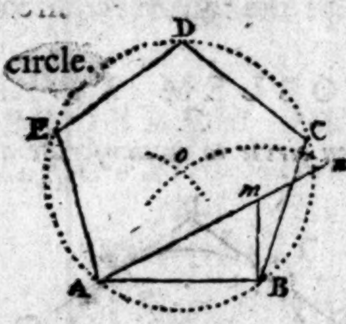
3. Through the points *n, m,* draw the lines A B, B C perpendicular to *on, om*; producing them till they meet each other at B.

4. In the ſame manner, draw the lines C D, D E, E A, and A B C D E will be the pentagon required.

Note. Any other polygon may be made to circumſcribe a circle, by firſt inſcribing a ſimilar one, and then drawing tangents to the circle at the angular points.

PROBLEM XXIX.*

On a given line A B to make a regular pentagon.



1. Make B *m* perpendicular to A B, and equal to one half of it.

---

* In the former edition of this work, another method of deſcribing a pentagon was given, as firſt propoſed by *Albertus Durer*, in his Geometry, p. 55, printed 1532; but as that is only an approximation, and is not more eaſy in practice than the preſent one, which is perfectly accurate, it is here omitted.

2. Draw A *m*, and produce it till the part *m n* is equal to B *m*.

3. From A and B as centres, with the radius B *n*, deſcribe arcs cutting each other in *o*.

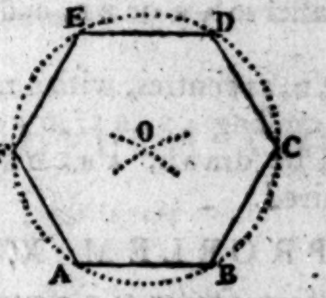
4. And from the point *o*, with the ſame radius, or with *o* A, or *o* B, deſcribe the circle A B C D E.

5. Apply the line A B five times round the circumference of this circle, and it will form the pentagon required.

Note. If tangents be drawn through the angular points A, B, C, D, E a pentagon circumſcribing the circle will be formed; and if the arcs be biſected, a circumſcribing decagon may be formed.

PROBLEM XXX.

On a given line A B to make a regular hexagon.



1. From the points A, B as centres, with the radius A B, deſcribe arcs cutting each other in *o*.

2. And from the point O, with the diſtance O A or O B, deſcribe the circle A B C D E F.

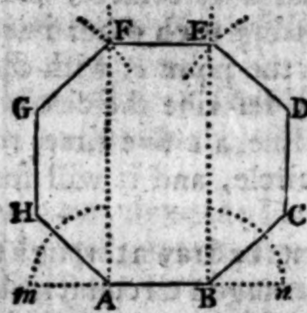
3. Apply the line A B ſix times round the circumference, and it will form the hexagon required *.

---

* This conſtruction is founded on the principle, that the radius of every circle is equal to the ſide of its inſcribed hexagon, or the chord of 60°.

PROBLEM XXXI.

On a given line A B *to form a regular octagon.*



1. On the extremes of the given line A B erect the indefinite perpendiculars A F and B E.

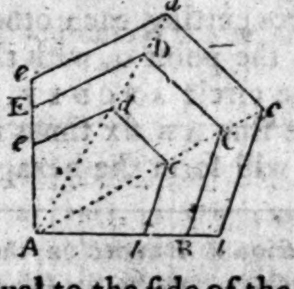
2. Produce A B both ways to *m* and *n*, and biſect the angles, *m* A F and *n* B E with the lines A H and B C.

3. Make A H and B C each equal to A B, and draw H G, C D parallel to A F or B E, and alſo each equal to A B.

4. From G, D, as centres, with a radius equal to A B, deſcribe arcs croſſing A F, B E, in F and E; and if G F, F E, and E D, be drawn, A B C D E F G H will be the octagon required.

PROBLEM XXXII.

To make a figure ſimilar to a given figure A B C D E.



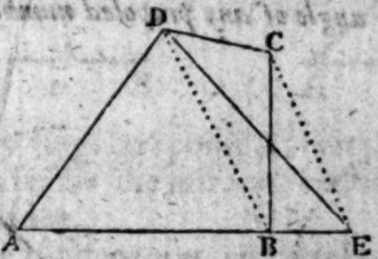
1. Take A *b* equal to the ſide of the figure required, and from the angle A draw the diagonals, A C, A D.

2. From the points *b*, *c*, *d* draw *bc*, *cd*, *de* parallel to B C, C D, D E, and A *bcde* will be ſimilar to A B C D E.

The ſame thing may alſo be done by making the angles *b*, *c*, *d*, *e* reſpectively equal to the angles B, C, D, E.

PROBLEM XXXIII.

To make a triangle equal to a given trapezium A B C D.

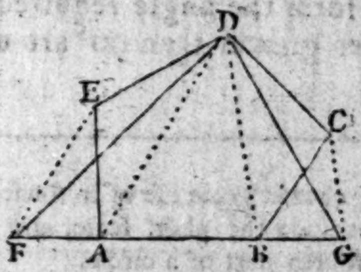


1. Draw the diagonal D B, and make C E parallel to it, meeting the ſide A B produced in E.

2. Join the points D, E, and A D E will be the triangle required.

PROBLEM XXXIV.

To make a triangle equal to any right lined figure A B C D E A.



1. Produce the ſide A B both ways at pleaſure.

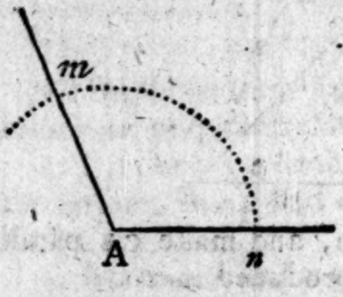
2. Draw the diagonals D A, D B, and parallel to them the lines E F and C G.

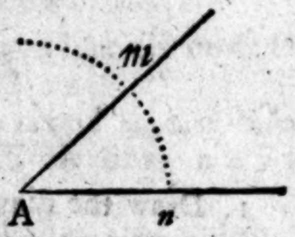
5. Join the points D F, D G, and D F G will be the triangle required.

And in nearly the ſame manner may any right lined figure whatever be reduced to a triangle.

PROBLEM XXXV.*

To make an angle of any propoſed number of degrees.





1. Take the firſt 60 degrees from the ſcale of chords, and from the point A, with this radius, deſcribe the arc *n m*.

2. Take the chord of the propoſed number of degrees from the ſame ſcale, and apply it from *n* to *m*.

3. From the point A draw the lines A*n* and A*m*, and they will form the angle required.

Note. Angles greater than 90° are uſually taken at twice.

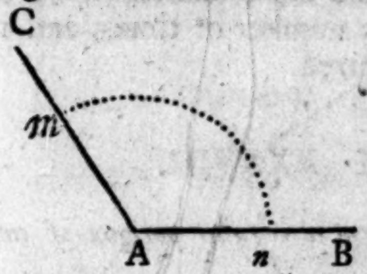
---

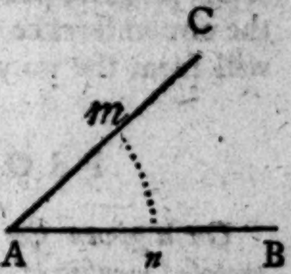
* The line of chords made uſe of in the following problems, is commonly put upon the plain ſcale, and is adapted to 90 degrees, or the fourth part of a circle.

For a deſcription of this, and other inſtruments made uſe of in Practical Geometry, ſee Mr. *Robertſon's Treatiſe on ſuch mathematical inſtruments as are uſually put into a portable caſe.*

PROBLEM XXXVI.

Any angle BAC *being given, to find the number of degrees it contains.**





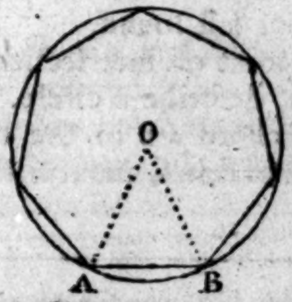
1. From the angular point A, with the chord of 60 degrees, deſcribe the arc *n m*, cutting the legs in the points *n* and *m*.

2. Take the diſtance *n m*, and apply it to the ſcale of chords, and it will ſhew the degrees required.

Note. When the diſtance *n m* is greater than 90°, it muſt be taken at twice, as before.

PROBLEM XXXVII.

In a given circle to inſcribe a polygon of any propoſed number of ſides.



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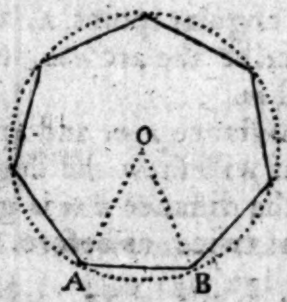
* Both this and the laſt problem may be performed by means of a *protractor*, which is a graduated arc deſigned for that purpoſe.

1. Divide 360° by the number of ſides, and make an angle A O B, at the centre, whoſe meaſure ſhall be equal to the degrees in the quotient.

2. Join the points A B, and apply the chord A B to the circumference the given number of times, and it will form the polygon required.

PROBLEM XXXVIII.

On a given line A B to form a regular polygon of any propoſed number of ſides.



1. Divide 360° by the number of ſides, and ſubtract the quotient from 180 degrees.

2. Make the angles A B O and B A O each equal to half the difference laſt found.

3. From the point of interſection O, with the diſtance O A or O B, deſcribe a circle.

4. Apply the chord A B to the circumference the propoſed number of times, and it will form the polygon required.*

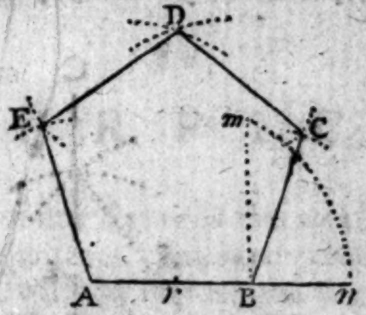
---

* By this method the circumference of a circle may alſo be divided into any number of equal parts; for if 360° be divided by the number of parts, and the angle A O B be made equal to the degrees in the quotient, the arc A B will be one of the equal parts required.

PRO-

PROBLEM XXXIX.*

Upon a given right line A B *to deſcribe a regular pentagon.*



1. Produce A B towards *n*, and at the point B make the perpendicular B *m* equal to A B.

2. Biſect A B in *r*, and from *r* as a centre, with the radius *r m*, deſcribe the arc *m n*, cutting A B in *n*.

3. From the points A and B, with the radius A *n*, deſcribe arcs cutting each other in D.

4. And from the points A, D and B, D, with the radius A B, deſcribe arcs cutting each other in C and E.

5. Join E C, D C, D E and E A, and A B C D E will be the pentagon required.

This method differs but little from that of Problem XXIX, and is equally eaſy and convenient in practice.

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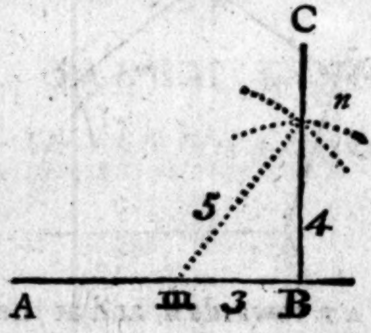
* This and the following problem were not given in the firſt Edition of this work, but are now added on account of their elegance and utility. The ſecond is derived from the 47th Prop. B. I. Euclid's Elements, and the firſt is propoſed for a demonſtration in the Ladies Diary for the year 1786.

PRO-

PROBLEM XL.

To raiſe a perpendicular from any point B *in a given line* AB.



1. From any ſcale of equal parts take a diſtance equal to 3 diviſions, and ſet it from B to *m*.

2. And from the points B and *m*, with the diſtances 4 and 5, taken from the ſame ſcale, deſcribe arcs cutting each other in *n*.

3. Through the points *n*, B, draw the line B C, and it will be the perpendicular required.

Explanation of the characters made uſe of in the following part of the Work.

$+$ Is the ſign of addition.
$-$ ——— of ſubtraction.
$\times$ ——— of multiplication.
$\div$ ——— of diviſion.
$\surd$ ——— of the ſquare root.
$^{3}\surd$ ——— of the cube root.
$=$ ——— of equality.
$: \; :: \; :$ ——— of proportion.

OF THE

MENSURATION

OF

SUPERFICIES.

THE *area* of any figure is the meaſure of its ſurface, or the ſpace contained within the bounds of that ſurface, without any regard to thickneſs.

A ſquare whoſe ſide is one inch, one foot, or one yard, &c. is called the *meaſuring unit*, and the area or content of any figure is computed by the number of thoſe ſquares contained in that figure.

PROBLEM I.

To find the area of a parallelogram; whether it be a ſquare, a rectangle, a rhombus, or a rhomboides.

RULE.*

Multiply the length by the perpendicular height, and the product will be the area.

EXAM-

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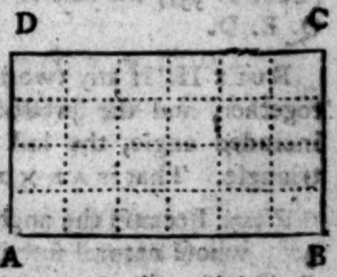
* Take any rectangle ABCD, and divide each of its ſides, reſpectively, into as many equal parts as is expreſſed by the number of times they contain the linear meaſuring unit, and let all the oppoſite points of diviſion be connected by right lines. Then, it is evident, that theſe lines divide the rectangle



EXAMPLES.

1. Required the area of the ſquare ABCD whoſe ſide is 5 feet 9 inches.

Here 5 *fe.* 9 *in.* $= 5.75$; *and* $\overline{5.75}|^{2} = 5.75 \times 5.75 =$ 33.0625 *feet* $=$ 33 *fe.* 0 *in.* 9 *pa.* $=$ *area required.*

---

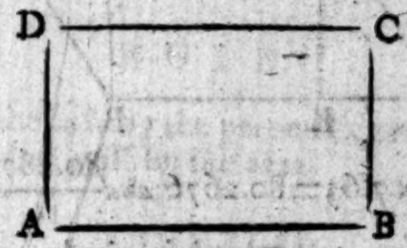
rectangle into a number of ſquares each equal to the ſuperficial meaſuring unit, and that the number of theſe ſquares, or the area of the figure, is equal to the number of linear meaſuring units in the length, as often repeated as there are linear meaſuring units in the breadth or height, that is equal to the length multiplied by the height, *which is the rule.*

And ſince a rectangle is equal to an oblique parallelogram ſtanding upon the ſame baſe, and between the ſame parallels, (Euc. I. 35,) the rule is true for any parallelogram in general. Q. E. D.

Rule II. If any two ſides of a parallelogram be multiplied together, and the product again by the natural ſine of their included angle, the laſt product will give the area of the triangle. That is AB × BC × nat. ſine of the angle B = area.

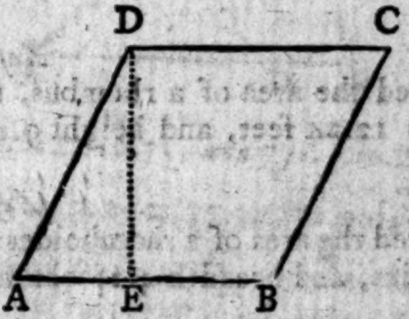
Note. Becauſe the angles of a ſquare and rectangle are each 90°, whoſe natural ſine is unity, or 1, the rule in this caſe is the ſame as that given in the text.

2. Required the area of the rectangle ABCD, whoſe length AB is 13.75 chains, and breadth BC 9.5 chains.



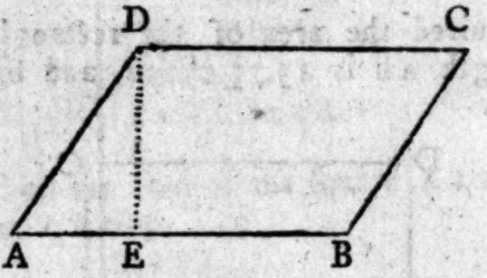
Here 13.75 × 9.5 = 130.625; *and* $\frac{130.625}{10}$ = 13.0625 *acres* = 13 *ac.* 0 *ro.* 10 *po.* = *area required.*

3. Required the area of the rhombus ABCD, whoſe length AB is 12 *fe.* 6 *in.* and its height DE 9 *fe.* 3 *in.*



Here 12 *fe.* 6 *in.* = 12.5, *and* 9 *fe.* 3 *in.* = 9.25; *Whence* 12.5 × 9.25 = 115.625 *feet.* = 115 *fe.* 7 *in.* 6 *pa.* = *area required.*

4. What is the area of the rhomboides ABCD, whoſe length AB is 10.52 chains, and height DE 7.63 chains?



Here 10.52 × 7.63 = 80.2676; *and* $\frac{80.2676}{10}$ = 80.2676 *acres* = 8 *ac.* 0 *ro.* 4 *po.* = *area required.*

5. What is the area of a ſquare whoſe ſide is 35.25 chains?

ac. ro. po.
Anſ. 124 1 1

6. What is the area of a ſquare whoſe ſide is 8 feet 4 inches?

fe. in. pa.
Anſ. 69 5 4

7. What is the area of a rectangle whoſe length is 14 feet 6 inches, and breadth 4 feet 9 inches?

fe. in. pa.
Anſ. 68 10 6

8. Required the area of a rhombus, the length of whoſe ſide is 12.24 feet, and height 9.16 feet.

fe. in. pa,
Anſ. 112 1 5

9. Required the area of a rhomboides whoſe length is 10.51 chains, and breadth 4.28 chains.

ac. ro. p.
Anſ. 4 1 39

10. What is the area of a rhomboides whoſe length is 7 feet 9 inches, and height 3 feet 6 inches?

fe. in. pa.
Anſ. 17 1 6

11. To find the area of a rectangular board, whoſe length is $12\frac{1}{2}$ feet, and breadth 9 inches.

Anſ. $9\frac{3}{8}$ *feet.*

PRO-

PROBLEM II.

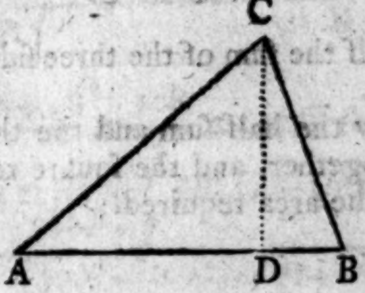
To find the area of a triangle.

RULE.*

Multiply the baſe by the perpendicular height, and half the product will be the area.

EXAMPLES.

1. Required the area of the triangle ABC, whoſe baſe AB is 10 feet 9 inches, and height DC 7 feet 3 inches.



Here 10 *fe.* 9 *in.* = 10.75, *and* 7 *fe.* 3 *in.* = 7.25;

Whence 10.75 × 7.25 = 77.9375, *and* $\frac{77.9375}{2}$ =

38.96875 *feet* = 38 *fe.* 11 *in.* $7\frac{1}{2}$ *pa.* = *area required.*

2. What is the area of a triangle whoſe baſe is 18 feet 4 inches, and height 11 feet 10 inches?

	fe.	*in.*	*pa.*
Anſ.	108	5	8

---

* A triangle is half a parallelogram of the ſame baſe and altitude, (Euc. I. 41.) and therefore the truth of the rule is evident.

3. What is the area of a triangle whoſe baſe is 16.75 feet, and height 6.24 feet? *fe. in. pa.*

Anſ. 52 3 1

4. Required the area of a triangle whoſe baſe is 12.25 chains, and height 8.5 chains. *ac. ro. po.*

Anſ. 5 0 33

5. What is the area of a triangle whoſe baſe is 20 feet, and height 10.25? *Anſ.* 102.5 *fe.*

PROBLEM III.

To find the area of a triangle whoſe three ſides only are given[*].

RULE.

1. From half the ſum of the three ſides ſubtract each ſide ſeverally.

2. Multiply the half ſum and the three remainders continually together, and the ſquare root of the product will be the area required.

EXAM-

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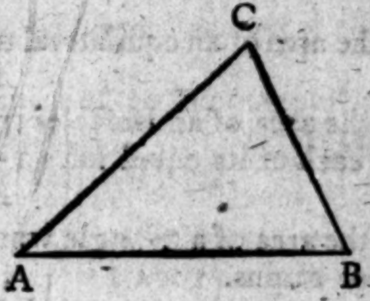
[*] *Demon.* Let $AC = a$, $AB = b$, $BC = c$, and $AD = x$: (See preceding fig.) Then, ſince $BD = b - x$, we ſhall have $c^2 - (b-x)^2 = CD^2 = a^2 - x^2$, or $c^2 - b^2 + 2bx - x^2 = a^2 - x^2$ from which x is found $= \frac{a^2+b^2-c^2}{2b}$ by tranſ. and reduction.

But $CD^2 = AC^2 - AD^2 = \overline{AC+AD} \times \overline{AC-AD} = (a + \frac{a^2+b^2-c^2}{2b}) \times (a - \frac{a^2+b^2-c^2}{2b}) = \frac{2ab+a^2+b^2-c^2}{2b} \times \frac{2ab-a^2-b^2+c^2}{2b} = \frac{(a+b)^2-c^2}{2b} \times \frac{c^2-(a-b)^2}{2b}$; Whence $CD = \frac{1}{2b}\sqrt{(\overline{a+b}^2 - c^2) \times (c^2 - \overline{a-b}^2)}$ and

the

EXAMPLES.

1. Required the area of the triangle A B C, whoſe three ſides B C, C A and A B, are 24, 36, and 48 chains reſpectively.



Here $\frac{24+36+48}{2}=\frac{108}{2}=54=\frac{1}{2}$ *ſum of the ſides;*

Alſo $54-24=30$ *firſt diff.* $54-36=18$ *ſecond diff. and* $54-48=6$ *third diff.*

Whence $\sqrt{54\times30\times18\times0}=\sqrt{174960}=418.043=$ *area required.*

2. Re-

---

the area $\frac{1}{2}\overline{AB}\times CD=\frac{1}{4}\sqrt{:(\overline{a+b}^2-c^2)\times(c^2-\overline{a-b}^2)}$ $=\frac{1}{4}\sqrt{(a+b+c\times\overline{a+b-c}\times\overline{c+a-b}\times\overline{c-a+b})}$ $=\sqrt{(\frac{a+b+c}{2}\times\frac{a+b-c}{2}\times\frac{c+a-b}{2}\times\frac{c-a+b}{2},)}$ which, by making $s=\frac{1}{2}\times(a+b+c)$ becomes $=\sqrt{(s\times\overline{s-c}\times\overline{s-b}\times\overline{s-a})}=$ algebraical expreſſion for the rule, as was to be demonſtrated.

Cor. 1. If s be put equal to $a+b$ and $d=b\backsim c$, the rule is $\sqrt{(s^2-a^2)\times(a^2-d^2.)}$

Cor.

2. Required the area of a triangle whoſe three ſides are 13, 14 and 15 feet. *Anſ.* 84 *ſquare feet.*

3. How many acres are there in a triangle whoſe three ſides are 49.00, 50.25 and 25.69 chains? *Anſ.* 61.498 *ac.*

4. Required the area of a right angled triangle, whoſe hypothenuſe is 50, and the other two ſides 30 and 40. *Anſ.* 600.

5. Required the area of an equilateral triangle whoſe ſide is 25. *Anſ.* 270.625.

6. Required the area of an Iſoſceles triangle whoſe baſe is 20, and each of its equal ſides 15. *Anſ.* 111.803.

7. Required the area of a triangle whoſe three ſides are 20, 30 and 40 chains. *Anſ.* 29 *ac.* 7 *po.*

PROBLEM IV.

Any two ſides of a right angled triangle being given to find the third ſide.

Cor. 2. If all the ſides be equal, the rule will become $\frac{1}{4}a^2\surd 3$, or $\frac{1}{4}a^2 \times 1.732$ for the equilateral triangle whoſe ſide is a.

Cor. 3. If the triangle be right angled, a being the hypothenuſe, the rule will be $\frac{a+b+c}{2} \times \frac{-a+b+c}{2}$, or $\frac{1}{2}p \times \overline{\frac{1}{2}p-a}$, putting p for the perimeter.

RULE II. Any two ſides of a triangle being multiplied together, and the product again by half the natural ſine of their included angle, will give the area of the triangle.

That is, AC × CB × nat. ſine of the angle C = twice area.

RULE.*

1. *When the two legs are given, to find the hypothenuſe.*

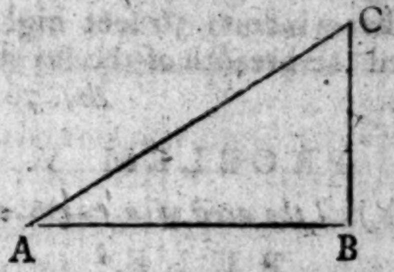
Add the ſquare of one of the legs to the ſquare of the other, and the ſquare root of the ſum will be equal to the hypothenuſe.

2. *When the hypothenuſe and one of the legs are given, to find the other leg.*

From the ſquare of the hypothenuſe take the ſquare of the given leg, and the ſquare root of the remainder will be equal to the other leg.

EXAMPLES.

1. In the right angled triangle ABC, the baſe AB is 56, and the perpendicular BC 33: what is the hypothenuſe?



Here $56^2+33^2=3136+1089=4225$; *and* $\sqrt{4225}=65=$ *hypothenuſe* AC.

2. If the hypothenuſe AC be 53, and the baſe AB 45: what is the perpendicular BC?

---

* By Euc. 47. I. $\overline{AB^2+BC^2}=AC^2$, or $\overline{AC^2-AB^2}=BC$; and therefore $\sqrt{\overline{AB^2+BC^2}}=AC$, or $\sqrt{\overline{AC^2-AB^2}}=BC$, or $\sqrt{\overline{AC^2-BC^2}}=AB$ which *is the ſame as the rule.*

Here $53^2-45^2=2809-2025=784$; *and* $\sqrt{784}=28$=*perpendicular* BC.

3. The baſe of a right angled triangle is 77, and the perpendicular 36: what is the hypothenuſe?

Anſ. 85.

4. The hypothenuſe of a right angled triangle is 109, and the perpendicular 60: what is the baſe?

Anſ. 91.

5. It is required to find the length of a ſhoar, which ſtrutting 12 feet from the upright of a building, will ſupport a jaumb 20 feet from the ground.

Anſ. 23.32380 *feet.*

6. The height of a precipice, ſtanding cloſe by the ſide of a river, is 103 feet, and a line of 320 feet will reach from the top of it to the oppoſite bank; required the breadth of the river. *Anſ.* 302.9703 *feet.*

7. A ladder 50 feet long, being placed in a ſtreet, reached a window 28 feet from the ground, on one ſide; and by turning it over, without removing the foot, it reached another window, 36 feet high on the other ſide: required the breadth of the ſtreet.

Anſ. 76.1233335 *feet.*

PROBLEM V.

To find the area of a trapezium.

RULE.*

Multiply the diagonal by the ſum of the two perpendiculars falling upon it from the oppoſite angles, and half the product will be the area.

EXAM-

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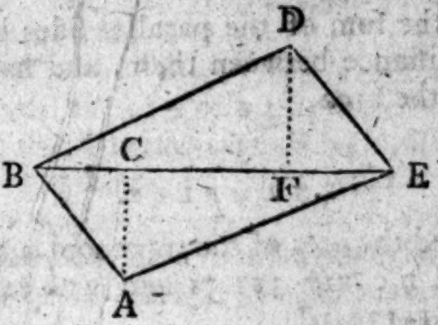
* *Demon.* The area of the triangle BDE is $= \frac{BE \times DF}{2}$; and the area of the triangle BAE is $= \frac{BE \times AC}{2}$; and

there

EXAMPLES.

1. Required the area of the trapezium BAED, whoſe diagonal BE is 84, the perpendicular AC 21, and DF 28.



Here $\overline{28+21}\times 84=49\times 84=4116$; *and* $\frac{4116}{2}=$ 2058 *the area required.*

2. Required the area of a trapezium whoſe diagonal is 80.5, and the two perpendiculars 24.5 and 30.1.
Anſ. 2197.65.

3. What is the area of a trapezium whoſe diagonal is 108 feet 6 inches, and the perpendiculars 56 feet 3 inches, and 60 feet 9 inches? *Anſ.* 6347 *fe.* 3 *in.*

---

therefore the ſum of theſe areas, or the area of the whole trapezium, is $=\frac{BE\times DF}{2}+\frac{BE\times AC}{2}=\frac{DF+AC}{2}\times BE$. Q.E.D.

If the trapezium can be inſcribed in a circle; that is, if the ſum of two of its oppoſite angles is equal to two right angles, or 180°, the area may be found thus:

Rule. From half the ſum of the four ſides ſubtract each ſide ſeverally; then multiply the four remainders continually together, and the ſquare root of the product will be the area.

PROBLEM VI.

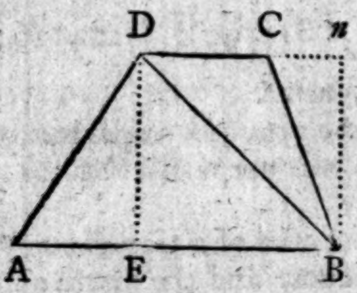
To find the area of a trapezoid, or a quadrangle, two of whoſe oppoſite ſides are parallel.

RULE.[*]

Multiply the ſum of the parallel ſides by the perpendicular diſtance between them, and half the product will be the area.

EXAMPLES.

1. Required the area of the trapezoid ABCD, whoſe ſides AB and DC are 321.51 and 214.24, and perpendicular DE 171.16.



Here 321.51 + 214.24 = 535.75 = *ſum of the parallel ſides* AB, DC.

---

[*] *Demon.* The △ ABD is $= \frac{AB \times DE}{2}$, and the △ BCD $= \frac{DC \times Bn}{2}$, or (becauſe $Bn = DE$) $= \frac{DC \times DE}{2}$. Therefore △ ABD + △ BCD, or the whole trapezoid ABCD, is $= \frac{AB \times DE}{2} + \frac{DC \times DE}{2} = \frac{AB + DC}{2} \times DE$. Q.E.D.

Whence 535.75 × 171.16 (*the perp.* DE) = 91698.9700;

And $\frac{91698.9700}{2}$ = 45849.485 *the area required.*

2. The parallel ſides of a trapezoid are 12.41 and 8.22 chains, and the perpendicular diſtance 5.15 chains; required the area. *ac. ro. po.*

Anſ. 5 1 9

3. Required the area of a trapezoid whoſe parallel ſides are 25 feet 6 inches and 18 feet 9 inches, and the perpendicular diſtance 10 feet 5 inches? *fe. in. pa.*

Anſ. 230 5 7

4. Required the area of a trapezoid whoſe parallel ſides are 20.5 and 12.25, and perpendicular diſtance 10.75. *Anſ.* 176.03125.

PROBLEM VII.

To find the area of a regular polygon.

RULE.*

Multiply half the perimeter of the figure by the perpendicular falling from its centre upon one of the ſides, and the product will be the area.

Note. The perimeter of any figure is the ſum of all its ſides.

EXAM-

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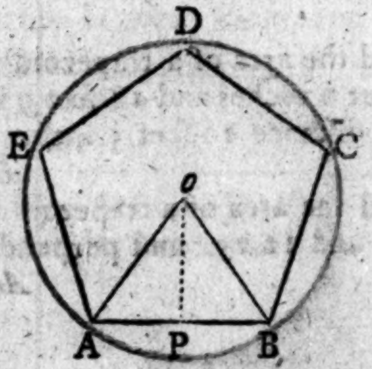
* *Demon.* Every regular polygon is compoſed of as many equal triangles as it has ſides, conſequently the area of one of thoſe triangles being multiplied by the number of ſides muſt give the area of the whole figure; but the area of either of the triangles is equal to the rectangle of the perpendicular and half the baſe, and therefore the rectangle of the perpendicular and

EXAMPLES.

1. Required the area of the regular pentagon ABCDE whoſe ſide AB, or BC, &c. is 25 feet, and perpendicular OP 17.2 feet?



Here $\frac{25\times5}{2}=62.5=$ *half perimeter; and* 62.5×17.2 $=1075$ *ſquare feet* $=$ *area required.*

2. Required the area of a hexagon whoſe ſide is 14.6 feet, and perpendicular 12.64 feet?

Anſ. 553.63[illegible] *ſquare feet.*

3. Required the area of a heptagon whoſe ſide is 19.38, and perpendicular from the centre 20?

Anſ. 1356.6.

4. Required the area of an octagon whoſe ſide is 9.941, and perpendicular 12? *Anſ.* 477.168.

---

and half the ſum of the ſides is equal to the area of the whole polygon: thus, OP × $\frac{AB}{2}$ is = area of the △ AOB, and OP × $\frac{5AB}{2}$ = area of the polygon ABCDE. Q. E. D.

PRO-

PROBLEM VIII.

To find the area of a regular polygon, when the ſide only is given.

RULE.*

Multiply the ſquare of the ſide of the polygon, by the number ſtanding oppoſite to its name in the following table, and the product will be the area.

N° of ſides.	Names.	Multipliers.
3	Trigon or equil. △	0.433013 —
4	Tetragon or ſquare	1.000000 +
5	Pentagon	1.720477 +
6	Hexagon	2.598076 +
7	Heptagon	3.633912 +
8	Octagon	4.828427 +
9	Nonagon	6.181824 +
10	Decagon	7.694209 —
11	Undecagon	9.365640 —
12	Duodecagon	11.196152 +

---

* *Demon.* The multipliers in the table are the areas of the polygons to which they belong when the ſide is unity or 1.

Whence as all regular polygons, of the ſame number of ſides, are ſimilar to each other, and as ſimilar figures are as the ſquares of their like ſides, (*Euc.* VI. 20.) 1^2 : multiplier in the table :: ſquare of the ſide of any polygon : area of that polygon; or, which is the ſame thing, the ſquare of the ſide of any polygon × by its tabular number is = area of the polygon. Q. E. D.

The table is formed by trigonometry thus: As radius = 1 : tang. $\angle$ OBP :: BP $(\frac{1}{2})$: PO $= \frac{\text{BP} \times \text{tang.} \angle \text{OBP}}{\text{radius.}} = \frac{1}{2}$ tang. $\angle$ OBP; whence OP × BP $= \frac{1}{4}$ tang. $\angle$ OBP = area

EXAMPLES.

1. Required the area of a pentagon whoſe ſide is 15.

$$\begin{array}{r}
15 \\
15 \\
\hline
75 \\
15 \\
\hline
225 = \textit{ſquare of the ſide.} \\
1.720477 = \textit{area, when the ſide is } 1. \\
225 \\
\hline
8602385 \\
3440954 \\
3440954 \\
\hline
387.107325 = \textit{area required.}
\end{array}$$

---

of the △ AOB; and ½ tang. ∠ OBP × number of ſides = tabular number, or the area of the polygon.

The angle OBP, together with its tangent, for any polygon of not more than 12 ſides, is ſhewn in the following table.

N° of ſides.	Names.	Angle OBP	Tangents.
3	Trigon	30°	$.57735+ = \frac{1}{3}\sqrt{3}$
4	Tetragon	45°	$1.00000+ = 1 \times 1$
5	Pentagon	54°	$1.37638+ = \sqrt{1+\frac{2}{5}\sqrt{5}}$
6	Hexagon	60°	$1.73205+ = \sqrt{3}$
7	Heptagon	$64^\circ\frac{2}{7}$	$2.07652+$
8	Octagon	$67^\circ\frac{1}{2}$	$2.41421+ = 1+\sqrt{2}$
9	Nonagon	70°	$2.74747+$
10	Decagon	72°	$3.07768+ = \sqrt{5+2\sqrt{5}}$
11	Undecagon	$73^\circ\frac{7}{11}$	$3.40568+$
12	Duodecagon	75°	$3.73205+ = 2+\sqrt{3}$

2. The ſide of a hexagon is 5 feet 4 inches; what is the area? *Anſ.* 73.9.

3. Required the area of an octagon whoſe ſide is 16? *Anſ.* 1236.0773.

4. Required the area of a decagon whoſe ſide is 20.5? *Anſ.* 3233.4913.

PROBLEM IX.

The diameter of a circle being given to find the circumference; or, the circumference being given to find the diameter.

RULE I.*

As 7 is to 22, ſo is the diameter to the circumference. Or, as 22 is to 7, ſo is the circumference to the diameter.

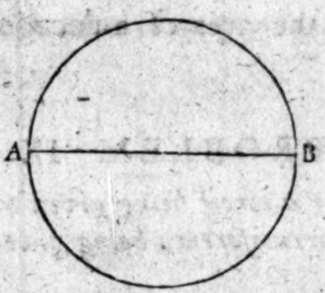
---

* The proportion of the diameter of a circle to its circumference has never yet been exactly attained. Nor can a ſquare, or any other right lined figure, be found, that ſhall be equal to a given circle. This is the celebrated problem called the *ſquaring of the circle,* which has exerciſed the abilities of the greateſt mathematicians for ages, and been the occaſion of ſo many endleſs diſputes. Several perſons of conſiderable eminence have, at different times, pretended that they had diſcovered the exact quadrature; but their errors have ſoon been detected, and it is now generally looked upon as a thing impoſſible to be done.

But though the relation between the diameter and circumference cannot be accurately expreſſed in known numbers, it may yet be approximated to any aſſigned degree of exactneſs. And in this manner was the problem ſolved by the great *Archimedes,* about two thouſand years ago, who diſcovered the proportion to be nearly as 7 is to 22, *which is the ſame as our firſt rule.* This he effected by ſhewing that the perimeter of a circumſcribed regular polygon of 192 ſides is to the diameter in a leſs

EXAMPLES.

1. If the diameter AB of a circle be 9, what is the circumference?



Here $7 : 22 :: 9 : 28\frac{2}{7}$; *Or* $\frac{22\times 9}{7} = 28\frac{2}{7} =$ *circumference required.*

2. If

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ratio than that of $3\frac{10}{70}$ to 1, and that the perimeter of an inſcribed polygon of 96 ſides is to the diameter in a greater ratio than that of $3\frac{10}{71}$ to 1, and from thence inferred the ratio abovementioned; as may be ſeen in his book *de dimenſione circuli.* The ſame proportion was alſo diſcovered by *Philo Gedarenſis,* and *Apollonius Pergæus,* at a ſtill earlier period, as we are informed by *Eutocius*, in his obſervations on a work not come to our hands, called *Ocyteboos.*

The proportion of *Vieta* and *Metius* is that of 113 to 355, which is ſomething more exact than the former, *and is the ſame as the ſecond rule.*

This is a very commodious proportion; for being reduced into decimals, it agrees with the truth as far as the ſixth figure incluſively. It was derived from the pretended quadrature of a M. Van-eick, which firſt gave riſe to the diſcovery.

But the firſt who aſcertained this ratio to any great degree of exactneſs was *Van Ceulen*, a *Dutchman,* in his book *de Circulo & Adſcriptis.* He found that if the diameter of a circle was 1, the circumference would be 3,141592653589793238462643383279502884 *nearly*; which is exactly true to 36 places

2. If the circumference of a circle ADBC be 36 feet, what is the diameter?

$$\begin{array}{l}
22 : 7 :: 36 \\
7 \\
\hline
22)252(11\tfrac{10}{22}\ \textit{feet, the diameter.} \\
22 \\
\hline
32 \\
22 \\
\hline
10
\end{array}$$

Or $\frac{36\times7}{22}=\frac{18\times7}{11}=\frac{126}{11}=11\tfrac{5}{11}$ *as before.*

RULE

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places of decimals, and was effected by means of the continual biſection of an arc of a circle, a method ſo exceedingly troubleſome and laborious, that it muſt have coſt him incredible pains. It is ſaid to have been thought ſo curious a performance, that the numbers were cut on his *tomb-ſtone*, in *St. Peter's Church-yard*, at *Leyden*. This laſt number has ſince been confirmed, and extended to double the number of places, by the late ingenious *Mr. Abraham Sharp*, of *Little Horton*, near *Bradford*, in *Yorkſhire*.

But ſince the invention of *Fluxions*, and the *Summation of Infinite Series*, there have been ſeveral methods found out for doing the ſame thing with much more eaſe and expedition. The late *Mr. John Machin*, *Profeſſor of Aſtronomy in Greſham College*, has, by theſe means, given a quadrature of the circle which is true to 100 places of decimals; and *M. De Lagny*, *M. Euler*, &c. have carried it ſtill farther. All of which proportions are ſo extremely near the truth, that except the ratio could be completely obtained, we need not wiſh for a greater degree of accuracy.

The method of obtaining this proportion from the doctrine of fluxions may be ſhewn as follows:

RULE II.

As 113 is to 355 ſo is the diameter to the circumference. Or, as 355 is to 113 ſo is the circumference to the diameter.

EXAMPLES.

1. The diameter of a circle is 9 feet; what is the circumference?

Here, as $113 : 355 :: 9 : 28\frac{31}{113}$.

$$Or\ \frac{355\times 9}{113}=\frac{3195}{113}=28.27=circumference.$$

2. If

Let $r=$radius, $x=$abciſſa, or verſed ſine, $y=$ordinate, or ſine, and $z=$length of the arc.

Then $\overline{2r-x}\times x=2rx-x^2=y^2$ by the property of the circle; from which x is found $=r+\sqrt{r^2-y^2}$, and $\dot{x}=-y\dot{y}\times\overline{r^2-y^2}|^{-\frac{1}{2}}$.

But $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$ as is ſhewn by the writers on fluxions; whence by ſubſtitution $\dot{z}=r\dot{y}\times\overline{r^2-y^2}|^{-\frac{1}{2}}$, which thrown into an infinite ſeries gives $\dot{z}=r\dot{y}\times(\frac{1}{r}+\frac{y^2}{2r^2}+\frac{3y^4}{8r^5}+\frac{5y^6}{16r^7}+\frac{35y^8}{128r^9}$ &c.); the fluent which is $z=y\times(1+\frac{y^2}{2.3r^2}+\frac{3y^4}{2.4.5r^4}+\frac{3.5y^6}{2.4.6.7r^6}+\frac{3.5.7y^8}{2.4.6.8.9r^8}$ &c. $=$length of the arc in terms of the ſine; which, when y is equal to r, becomes $r\times(1+\frac{1}{2.3}+\frac{3}{2.4.5}+\frac{3.5}{2.4.6.7}+\frac{3.5.7}{2.4.6.8.9}$ &c. $=\frac{1}{4}$ of the circumference.

And if inſtead of y in this ſeries be ſubſtituted its value $\frac{rt}{\sqrt{r^2+t^2}}$, we ſhall have $z=t-\frac{t^3}{3r^2}+\frac{t^5}{5t^4}-\frac{t^7}{7r^6}+\frac{t^9}{9r^8}$ &c. $=$ length of the arc in terms of the tangent. Where, by taking the arc ſo that the tangent can be found in terms of the radius, the ſeries will become known, and may be determined to any degree of exactneſs.

Thus,

2. If the diameter of a circle be 10 feet, what is the circumference?

As 113 : 355 : : 10

$$\begin{array}{r@{}l}
 & 355 \\
 & \underline{10} \\
113) & 3550(31\tfrac{47}{113}, \textit{ feet the circumference.} \\
 & \underline{339} \\
 & 160 \\
 & \underline{113} \\
 & 47
\end{array}$$

Or $\frac{355 \times 10}{113} = \frac{3550}{113} = 31.41 =$ *circumference.*

3. What

Thus, if the radius be 1, and the arc be $\frac{1}{8}$ part of the circumference, or 45°, its tangent will be equal to the radius, and the ſeries will become $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$ &c. = arc of 45°, and $8 \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$ &c.) = whole circumference.

This ſeries is the ſimpleſt form that can poſſibly be obtained, but in order to get another that will converge faſter, we muſt take a ſmaller arc; as for inſtance, ſuppoſe that of 30°, or $\frac{1}{12}$ part of the circumference.

Then ſince the tangent of 30°, to radius 1, is $\surd\frac{1}{3}$, the general ſeries will become $\surd\frac{1}{3} \times (1 - \frac{1}{3\cdot3} + \frac{1}{5\cdot3^2} - \frac{1}{7\cdot3^3} + \frac{1}{9\cdot3^4}$ &c.) = arc of 30°; and $12\surd\frac{1}{3} \times (1 - \frac{1}{3\cdot3} + \frac{1}{5\cdot3^2} - \frac{1}{7\cdot3^3} + \frac{1}{9\cdot3^4}$ &c.) = whole circumference; *and ſo for any other arc whatever.*

Those

3. What is the diameter of a circle whoſe circumference is 50?

As 355 : 113 :: 50
50

355)5650(15 *the diameter.*
355

2100
1775

325

RULE III.

Multiply the diameter by 3.1416, and the product will be the circumference; or

Divide the circumference by 3.1416, and the quotient will be the diameter.

EXAMPLES.

1. If the diameter of a circle be 17, what is the circumference?

3.1416
17

219912
31416

53.4072=*circumference.*

Thoſe who would wiſh to ſee the methods of *Machin*, *Euler*, &c. may conſult Dr. Hutton's Menſuration, and a paper of his in the Philoſophical Tranſactions upon this ſubject.

2. If the circumference of a circle be 354, what is the diameter?

By Rule I.

As $22 : 7 :: 354 : \frac{354 \times 7}{22} = 122.639 =$ *circumference.*

By Rule II.

As $355 : 113 :: 354 : \frac{354 \times 113}{355} = 112.681 =$ *circumference.*

By Rule III.

As $3.1416 : 1 :: 354 : \frac{354}{3.1416} = 112.681 =$ *circumference, agreeing with the former as far as the third place of decimals.*

3. What is the circumference of a circle whoſe diameter is 40 feet? *Anſ.* 125.6640.

4. If the circumference of a circle be 12, what is the diameter? *Anſ.* 3.81972.

5. The earth is a globe, and its circumference is known to be 25000 miles, what is its diameter? *Anſ.* 7958 *nearly.*

If d be put = diameter of any circle, and c = circumference, the three rules above given, may be expreſſed thus:

$$\frac{22d}{7} = c;\quad \frac{355d}{113} = c;\quad 3.1416d = c,$$

$$\frac{7c}{22} = d;\quad \frac{113c}{355} = d;\quad \frac{c}{3.1416} = d.$$

And if r, or the radius, be uſed inſtead of the diameter, theſe rules become

$$\frac{44r}{7} = c;\quad \frac{710r}{113} = c;\quad 6.2832r = c,$$

$$\frac{7c}{44} = r;\quad \frac{113c}{710} = r;\quad \frac{c}{6.2832} = r.$$

PROBLEM X.

To find the length of any arc of a circle.

RULE I.*

From 8 times the chord of half the arc ſubtract the chord of the whole arc, and $\frac{1}{3}$ of the remainder will be the length of the arc *nearly*.

EXAMPLES.

1. The chord of the whole arc DE is 48, and the verſed ſine BC of half the arc is 18: what is the length of the arc DCE?

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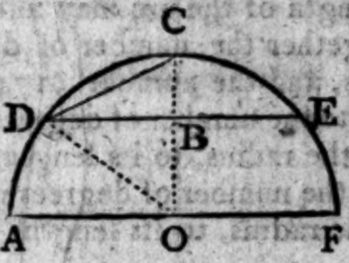
* *Demon.* Let the radius OD $= r$, and ſine DB $= s$. Then will the chord DC $= \sqrt{s^2 + (r - \sqrt{r^2 - s^2})^2} = s + \frac{s^3}{8r^2} + \frac{7s^5}{128r^4}$ &c. Whence 8 times the chord DC $= 8s + \frac{s^3}{r^2} + \frac{7s^5}{16r^4}$ &c. And 8 DC $-$ DE $= 6s. + \frac{s^3}{r^2} + \frac{7s^5}{16r^4}$ &c. whoſe $\frac{1}{2}$ part is $2s + \frac{s^3}{3r^2} + \frac{7s^5}{48r^4}$ &c.

But the length of the arc DC, whoſe ſine is s, is known to be $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4}$ &c. and thereſore the arc DE $= 2s + \frac{s^3}{3r^2} + \frac{6s^5}{40r^4}$ &c. which differs from $2s + \frac{s^3}{3r^2} + \frac{7s^5}{48r^4}$ &c. only by a ſmall quantity, and ſhews the rule to be very near the truth. Q. E. D.

Rule 2. If d be put $=$ to the diameter, and $v =$ verſed ſine of half the arc, then will $(5d\sqrt{\frac{5d}{5d-3v}} + 4\sqrt{dv}) \times \frac{2}{9} =$ length of the arc *nearly*.

Here



Here $24^2+18^2=576+324=900$; *and* $\sqrt{900}=30=$ DC. *Whence* $\frac{30\times 8-48}{3}=\frac{240-48}{3}=\frac{192}{3}=64$ *the length of the arc required.*

2. The chord of the whole arc is 50.8, and the chord of half the arc is 30.6: required the length of the arc. *Anſ.* 64.6.

3. The length of the whole chord is 6, and the length of the chord of half the arc 3.0538: what is the length of the arc? *Anſ.* 6.14768.

RULE II.*

1. Add the ſquare of half the chord to the ſquare of the verſed ſine of half the arc, and this ſum divided by the verſed ſine will give the diameter.

2. Take $\frac{41}{50}$ of the verſed ſine from the diameter, and divide $\frac{2}{3}$ of the verſed ſine by the remainder.

3. To the quotient, laſt found, add 1; and this ſum multiplied by the chord of the whole arc, will give the length of the arc *nearly*.

Note.

---

* *Demon.* Let the diameter AF $=d$, the verſed ſine CB $=v$, and the chord DE $=c$.

Then

Note. The length of the arc may alſo be found by multiplying together the number of degrees it contains, the radius, and the number .01745329.

Or, as 180 is to the number of degrees in the arc, ſo is 3.1416 times the radius, to its length.

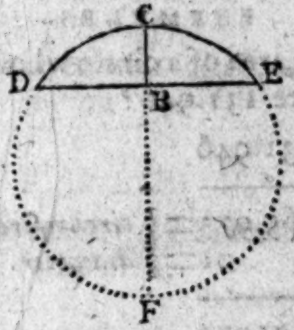
Or, as 3 is to the number of degrees in the arc, ſo is .05236 times the radius, to its length.

EXAMPLES.

1. If the chord DE be 48, and the verſed ſine CB 18: what is the length of the arc?

Then will the rule be expreſſed by $(\frac{2}{3}v \div \overline{d-\frac{41}{50}v}+1) \times c$ or by $(\frac{2}{3}v \div \overline{d-\frac{41}{50}v}+1) \times 2\sqrt{dv-v^2}$, where $d = \frac{c^2}{40}+v$

But $\frac{\frac{2}{3}v}{d-\frac{41}{50}v}+1 = 1+\frac{2v}{3d}+\frac{41v^2}{75d^2}+\frac{41\times 41v^3}{75\times 50d^3}$ &c. and $2\sqrt{dv-v^2} = 2\sqrt{dv} \times : 1-\frac{v}{2d}-\frac{v^2}{8d^2}$ &c. Whence $(\frac{2}{3}v \div \overline{d-\frac{41}{50}v}+1) \times 2\sqrt{dv-v^2} = (1+\frac{2v}{3d}+\frac{41v^2}{75d^2}$ &c). $\times\ (2\sqrt{dv} \times : 1-\frac{v}{2d}-\frac{v^2}{8d^2}$ &c.) $= 2\sqrt{dv} \times : 1+\frac{v}{6d}+\frac{53v^2}{600d^2}$ &c. Now $2\sqrt{dv} \times : 1+\frac{v}{6d}+\frac{3v^2}{40d^2}$ &c. is known to be the length of an arc whoſe diameter is d, and the verſed ſine of the half thereof v; and this differs from $2\sqrt{dv} \times : +\frac{v}{6d}+\frac{53v^2}{600d^2}$ &c. in the third term only by $\frac{v^2}{75d^2}$, which ſhews the rule to be near an approximation. Q. E. I.



Here $\frac{24^2+18^2}{18} = \frac{576+324}{18} = \frac{900}{18} = 50 =$ *diameter* CF.

And $18 \times \frac{2}{3} \div (50 - \frac{41}{50} \times 18) = 12 \div \overline{50 - 14.70} = 12 \div 35.24 = .3405.$

Whence $(1 + .3405) \times 48 = 1.3405 \times 48 = 64.3440 =$ *length of the arc required.*

2. The chord of the whole arc is 7, and the verſed ſine 2: what is the length of the arc?

Anſ. 8.8439.

3. The chord of the whole arc is 40, and the verſed ſine 15: what is the length of the arc?

Anſ. 53.62.

PROBLEM XI.

To find the area of a circle.

RULE I.*

Multiply half the circumference by half the diameter, and the product will be the area.

Or take $\frac{1}{4}$ of the product of the whole circumference and diameter.

EXAM-

---

* *Demon.* A circle may be conſidered as a regular polygon of an infinite number of ſides, the circumference being equal to

EXAMPLES.

1. What is the area of a circle whoſe diameter is 42, and circumference 131.946?

2)131.946

65.973=½ *circumference.*
21=½ *diameter.*

65973
131946

1385.433=*area required.*

2. What is the area of a circle whoſe diameter is 10 feet 6 inches, and circumference 31 feet 6 inches?

ft. *in.*
15 9=15.75=½ *circumference.*
5 3= 5.25=½ *diameter.*

7875
3150
7875

82.6875
12

8.2500

Anſ. 82 *feet*, 8 *inches.*

---

to the perimeter, and the radius to the perpendicular. But the area of a regular polygon is equal to half the perimeter multiplied by the perpendicular, and conſequently the area of a circle is equal to half the circumference multiplied by the radius, or half the diameter. Q. E. D.

This rule may be otherwiſe demonſtrated by the doctrine of fluxions.

3. What is the area of a circle whoſe diameter is 1, and circumference 3.14159? *Anſ.* .7854.

4. What is the area of a circle whoſe diameter is 7, and circumference 22? *Anſ.* $38\frac{1}{2}$.

RULE II.*

Multiply the ſquare of the diameter by .7854, and the product will be the area.

EXAM-

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* *Demon.* All circles are to each other as the ſquares of their diameters. (Euc. XII. 2.)

But the area of a circle whoſe diameter is 1, is .7854 &c. (by Rule I.) Therefore $1^2 : d^2 :: .7854\ \&c. : \frac{.7854\ \&c. \times d^2}{1}$ $= .7854\ \&c. \times d^2 =$ area of a circle whoſe diameter is *d*.

Q. E. D.

The following proportions are thoſe of *Metius* and *Archimedes*.

As 432 : 355 : : ſquare of the diameter : Area.
As 14 : 11 : : ſquare of the diameter : Area.

If the circumference be given, inſtead of the diameter, the area may be found as follows:

The ſquare of the circumference × .07958 = Area.
As 88 : 7 : : ſquare of the circumference : Area.
As 1420 : 113 : : ſquare of the circumference : Area.

And if *d* be the diameter, *c* the circumference, *a* the area, and $p = 3.14159$ &c. then:

$$1.\ d = \frac{c}{p} = \frac{4a}{c} = 2\sqrt{\frac{a}{p}}$$

$$2.\ c = pd = \frac{4a}{d} = 2\sqrt{pa}$$

$$3.\ a = \frac{pd}{4} = \frac{c}{4p} = \frac{dc}{4}$$

The

EXAMPLES.

1. What is the area of a circle whoſe diameter is 5?

.7854
25 = *ſquare of the diameter.*

39270
15708

19.6350 = *the anſwer.*

2. What is the area of a circle whoſe diameter is 7? *Anſ.* 38.4846.

3. What is the area of a circle whoſe diameter is 4.5? *Anſ.* 15.90435.

4. How many ſquare yards are there in a circle whoſe diameter is 3½ feet? *Anſ.* 1.069016.

PROBLEM XII.

To find the area of a ſector, or that part of a circle which is bounded by any two radii and their included arc.

The following table will alſo ſhew moſt of the uſeful problems, relating to the circle and its equal or inſcribed ſquare.

1. diameter × .8862 = ſide of an equal ſquare.
2. circumf. × .2821 = ſide of an equal ſquare.
3. diameter × .7071 = ſide of the inſcribed ſquare.
4. circumf. × .2251 = ſide of the inſcribed ſquare.
5. area × .6366 = ſide of the inſcribed ſquare.
6. ſide of a ſquare × 1.4142 = diameter of its circumf. circle.
7. ſide of a ſquare × 4.443 = circumf. of its circumf. circle.
8. ſide of a ſquare × 1.128 = diameter of an equal circle.
9. ſide of a ſquare × 3.545 = circumf. of an equal circle.

RULE.

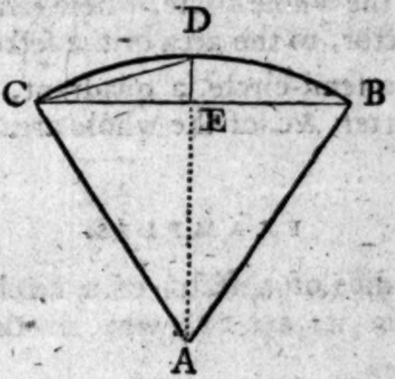
RULE.*

Multiply the radius, or half the diameter, by half the length of the arc of the ſector, and the product will be the area.

Note. The length of the arc may be found by either of the laſt problems.

EXAMPLES.

1. The radius AB is 40, the chord BC of the whole arc 50, and the chord CD of half the arc 27; required the area of the ſector.



Here $\frac{27\times8-50}{3}=\frac{216-50}{3}=\frac{166}{3}=55\cdot\dot{3}=$ *length of the arc* CDB.

And $\frac{55\cdot\dot{3}}{2}\times40=27.\dot{6}\times40=1109.\dot{6}=$ *area of the ſector required.*

---

* The rule for finding the area of the ſector, is, evidently, the ſame as that for finding the area of the whole circle.

2. Required the area of a ſector of a circle, whoſe radius is 25, and the length of the arc 21.5.

Anſ. 268.75.

3. What is the area of a ſector whoſe radius is 10, the chord of the whole arc 20, and the chord of half the arc 11? *Anſ.* 113.3.

4. Find the area of the ſector whoſe radius is 9, and the chord of whoſe arc is 6.

Anſ. 27.5267835.

RULE II.*

As 360 is to the degrees in the arc of a ſector, ſo is the area of the whole circle, whoſe radius is equal to that of the ſector, to the area of the ſector required.

Note. For a ſemi-circle, a quadrant, &c. take one half, one quarter, &c. of the whole area.

EXAMPLES.

1. The radius of a ſector of a circle is 20, and the degrees in its arc 22; what is the area of the ſector?

---

* *Demon.* Let $r=$ radius, $d=$ number of degrees in the arc of the ſector, and $A=$ its area.

Then will $4r^2\times .7854=r^2\times 3.1416=$ area of the whole circle, and $2r\times 3.1416=$ its circumference.

Alſo $360 : 2r\times 3.1416 :: d : \frac{2dr\times 3.1416}{360}=$ length of the arc of the ſector. But $\frac{2dr\times 3.1416}{360}\times\frac{1}{2}\times r=\frac{dr^2\times 3.1416}{360}=A$, by the laſt rule. And conſequently $360 : d :: r^2\times 3.1416 : A$.

Q. E. D.

.7854
1600 = *ſquare of the diameter.*

4712400
7854

360 : 22 :: 1256.6400 = *area of the whole circ*[illegible]
22

251328
251328

36.0)2764.608 (7.6 = *area of the ſector.*
252

244
216

286
&c.

2. Required the area of a ſector whoſe radius is 25, and the length of its arc 147 degrees 29 minutes.
Anſ. 804.4017.

3. Required the area of a ſemicircle whoſe radius is 13. *Anſ.* 265.4652.

4. Required the area of a quadrant whoſe radius is 21. *Anſ.* 346.3614.

PROBLEM XIII.

To find the area of a ſegment of a circle.

RULE I.

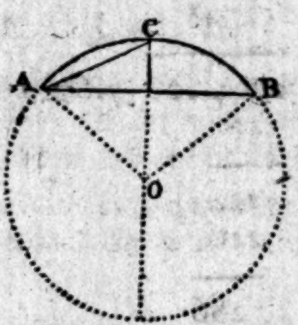
1. Find the area of the ſector, having the ſame arc with the ſegment, by one of the laſt problems.

2. Find the area of the triangle formed by the chord of the ſegment, and the radii of the ſector.

3. Then the ſum, or difference, of theſe areas, according as the ſegment is greater or leſs than a ſemicircle, will be the area required.

EXAMPLES.

1. The radius OB is 10, the chord AB is 18, and the chord AC 10; what is the area of the ſegment ABC?



Here $\frac{10\times8-18}{3}=\frac{80-18}{3}=\frac{62}{3}=20.6=$ *arc* ACB.

And $\frac{20.\dot{6}}{2}\times10=10.\dot{3}\times10=103.\dot{3}=$ *area of the ſector* OACB.

Alſo $\frac{18+10+10}{2}=\frac{38}{2}=19=\frac{1}{2}$ *ſum of the ſides* AO, OB, BA.

Whence $19-10=9=$ 1ſt *and* 2d *diff. and* $19-18=1=$ 3d *diff.*

And $\sqrt{19\times9\times9\times1}=\sqrt{1539}=39.23=$ *area of the* $\triangle$ BOA.

And $103.\dot{3}-39.23=64.10=$ *area of the ſegment required.*

It being in this caſe leſs than a ſemicircle.

2. Required the area of a ſegment whoſe height is 2, and chord 20. *Anſ.* 26.878.

3. Required the area of a ſegment of a circle whoſe radius is 24, the chord of the whole arc 20, and the chord of half the arc 10.2. *Anſ.* 28.07.

4. Required the area of a ſegment of a circle whoſe chord is 12, and the radius of the circle 10.
Anſ. 16.3274.

5. Required the area of a ſegment of a circle whoſe chord is 16, and the diameter of the circle 20.
Anſ. 44.7292.

6. What is the area of a ſegment, whoſe arc is a quadrant, the diameter being 18 feet? *Anſ.* 63.1174.

7. What is the area of a ſegment of a circle whoſe arc contains 280 degrees, the diameter being 50?
Anſ. 1834.9191.

RULE II.*

1. Add the ſquare of half the chord of the ſegment to the ſquare of its height, and multiply the ſquare root of the ſum by 4.

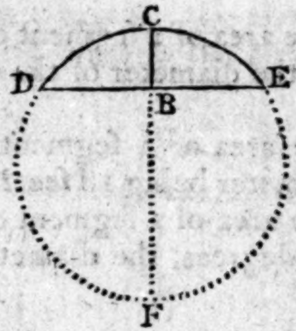
---

* *Demon.* Let $c =$ BE, $v =$ CB, and $d =$ diameter CF; then will the rule be expreſſed by $(2c + \frac{4}{3}\sqrt{c^2 + v^2}) \times \frac{2}{5}v$. But $(2c + \frac{4}{3}\sqrt{c^2 + v^2})\ \frac{2}{5}v = (\sqrt{dv - v^2} + \frac{2}{5}\sqrt{dv}) \times \frac{4}{5}v = \frac{4}{5}v \times \sqrt{dv} + \frac{4}{5}v\sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{8d^2} - \frac{3v^3}{48d^3}$ &c.$) = \frac{4}{3}v\sqrt{dv} - \frac{2v^2\sqrt{dv}}{5d} - \frac{2v^3\sqrt{dv}}{20d^2}$ &c. $= 2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{20d^2} - \frac{v^3}{72d^3}$ &c.) And ſince BB $= \sqrt{dv - v^2}$, and CB $= v$, therefore $\dot{v}\sqrt{dv} \times : 1 - \frac{v}{2d} - \frac{v^2}{8d^2} - \frac{3v^3}{48d^3}$ &c. will be the fluxion of the ſegment; and its fluent, $v\sqrt{dv}$ ×

2. To $\frac{1}{3}$ of the number laſt found, add the whole chord of the ſegment, and this ſum multiplied by $\frac{2}{5}$ of the height will give the area required.

EXAMPLES.

1. The chord DE of the ſegment DEC is 40, and its height, or verſed ſine, BC is 10: what is the area of the ſegment?



Here $4\sqrt{20^2+10^2}=4\sqrt{400+100}=4\sqrt{500}=4\times 22.36=89.44$.

And $\left(\frac{89.44}{3}+40\right)\times\frac{2\times 10}{5}=\overline{29.81+40}\times 4=69.81\times 4=279.24=$ *area required.*

$\times : \frac{2}{3}-\frac{v}{5d}-\frac{v^2}{8d^2}$ &c. will be the value of half the ſegment. Conſequently its double, $2v\sqrt{dv}\times\left(\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}-\frac{v^3}{72d^3}\right.$ &c.$\left.\right)$ will be the value of the whole ſegment; which differs from the expreſſion for the rule only in the third term, and therefore ſhews the *approximation* required. Q. E. D.

This rule was firſt given by *Sir Iſaac Newton*, and publiſhed by *Jones*, *Wallis*, and *Colſon*.

If $c=$ chord of the ſegment, and $v=$ its height; then $\left(7\sqrt{\frac{1}{4}c^2+\frac{2}{7}v^2}+\frac{4}{3}\sqrt{\frac{1}{4}c^2+v^2}\right)\times\frac{4}{25}v=$ area *very nearly*, which is another approximating rule, ſufficiently eaſy in its application.

2. The chord is 20, and the height, or verſed ſine 2; required the area of the ſegment.

Anſ. 26.85908.

3. The length of the chord is 48, and the height of the ſegment 18; what is the area? *Anſ.* 633.6.

RULE III.*

1. Divide the height, or verſed ſine, by the diameter, and find the quotient in the table of verſed ſines.

2. Mul-

---

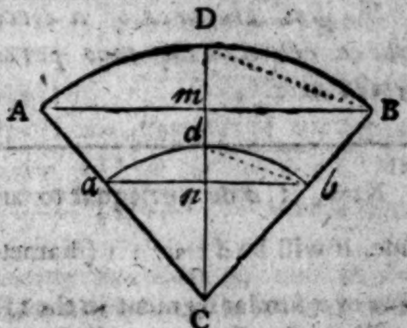
* The table to which this rule refers, is formed of the areas of the ſegments of a circle whoſe diameter is 1; and which is ſuppoſed to be divided by perpendicular chords into 1000 equal parts.

The reaſon of the rule itſelf depends upon this property—That the verſed ſines of ſimilar ſegments are as the diameters of the circles to which they belong, and the areas of thoſe ſegments as the ſquares of the diameters; which may be thus demonſtrated.

Let ADBA and *adba* be any two ſimilar ſegments, cut off from the ſimilar ſectors ADBCA and *adbca*, by the chords AB and *ab*; and let the perpendicular CD biſect them.

Then, by ſimilar triangles, $DB : db :: BC : bc$, and $DB : db :: Dm : dn$; whence, by equality, $BC : bc :: Dm : dn$, or $2BC : 2bc :: Dm : dn$.

Again, ſince ſimilar ſegments are as the ſquares of their chords, it will be $AB^2 : ab^2 ::$ ſeg. ADBA : ſeg. *adba*; but $AB^2 : ab^2 :: CB^2 : cb^2$, whence, by equality, ſeg. ADBA : ſeg. *adba* $:: CB^2 : cb^2$, or ſeg. ADBA : ſeg. *adba* $:: 4CB^2 : 4cb^2$. Q.E.D.



Now,

2. Multiply the number on the right hand of the verſed ſine, by the ſquare of the diameter, and the product will be the area.

EXAMPLES.

1. If the chord of a circular ſegment be 40, its verſed ſine 10, and the diameter of the circle 50: what is the area?

5.0)1.0

.2 = *tabular verſed ſine.*
.111823 = *tabular ſegment.*
2500 = *ſquare of* 50.

55911500
223646

279.557500 = *area required.*

2. The chord of the ſegment is 20, the verſed ſine 5, and the diameter 25: what is the area?

Anſ. 69.889375.

PROBLEM XIV.

To find the area of a circular zone, or the ſpace included between any two parallel chords and their intercepted arcs.

Now, if d be put equal to any diameter, and v the verſed ſine, it will be $d : v :: 1$ (diameter in the table) $: \frac{v}{d} =$ verſed ſine of a ſimilar ſegment in the table, whoſe area let be called a.

Then $1^2 : d^2 :: a : ad^2 =$ area of the ſegment whoſe height is v, and diameter d, as in the rule.

RULE.

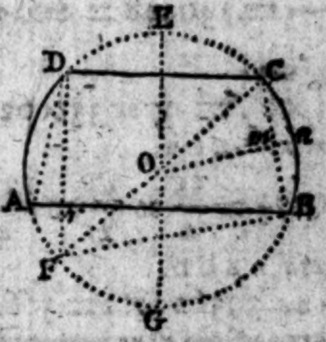
RULE.*

1. Find the area of that part of the zone which forms a trapezoid by problem 6, and the area of the ſegment B*n*C B by problem 13, rule 3.

2. Add the area of the trapezoid to twice the area of the ſegment, and it will give the area of the zone required.

EXAMPLES.

1. The greater chord A B is 20, the leſs D C 15, and their diſtance D*r* $17\frac{1}{2}$: required the area of the zone A B C D.



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* The reaſon of this rule is too obvious to require a demonſtration.

Note. When the two parallel ſides of the zone are equal, the chord of the ſmall ſegment will be equal to the breadth of the zone, and its height will be equal to $\sqrt{\frac{1}{4}AB^2+\frac{1}{4}as^2}-\frac{1}{2}AB$.

And when one of the ſides is the diameter of the circle, the chord of the ſame ſegment will be $\sqrt{as^2+D^2}$, and its height $=\frac{1}{2}AB-\sqrt{\frac{1}{4}AB^2-\frac{1}{4}as^2-\frac{1}{4}D^2}$ where $D=AB-DC$.

Another method, is by finding the areas of each of the ſegments A B E A, D C E D, the difference of which is the area of the zone required; but this rule, when the ſegments are large, and any of the approximating theorems are uſed, is not ſo accurate as the former.

Here $\sqrt{\mathrm{D}r^2+r\mathrm{A}^2}=\sqrt{17.5^2+2.5^2}=\sqrt{306.25+6.25}=\sqrt{312.5}=17.6776=\mathrm{AD}$ *or* CB.

And $\mathrm{D}r\ (17\tfrac{1}{2}) : r\mathrm{A}\ (2\tfrac{1}{2}) :: \mathrm{B}r\ (17\tfrac{1}{2}) : 2\tfrac{1}{2}=r\mathrm{F}$; *or* $2\tfrac{1}{2}+17\tfrac{1}{2}=20=\mathrm{DF}$.

Alſo $\sqrt{\mathrm{DF}^2+\mathrm{DC}^2}=\sqrt{20^2+15^2}=\sqrt{400+225}=\sqrt{625}=25=\mathrm{CF}$; *or* $\frac{25}{2}=12\tfrac{1}{2}=\mathrm{O\,C}$ *or* $\mathrm{O}n$.

And $r\mathrm{D}\ (17\tfrac{1}{2}) : \mathrm{AD}\ (17.6776) :: r\mathrm{B}\ (17\tfrac{1}{2}) : 17.6776=\mathrm{FB}$; *or* $\frac{17.6776}{2}=8.8388=\mathrm{O}m$; *and* $12.5-8.8388=3.6612=mn=$ *height of the ſegment.*

Therefore $\frac{3.6612}{25}=.146448=$ *tab. verſed ſine*; *and* $071033=$ *tab. ſegment.*

Whence $.071033\times25^2=.071033\times625=44.395625=$ *area of the ſegment* B*n*CB.

Alſo $\frac{15+20}{2}\times17\tfrac{1}{2}=\frac{35}{2}\times\frac{35}{2}=\frac{1225}{4}=306.25=$ *area of the trapezoid* ABCD.

And $306.25+44.395625\times2=306.25+88.79025=394.04025=$ *area of the zone required.*

2. The greater ſide is 96, the leſſer 60, and the breadth $17\tfrac{1}{2}$: what is the area of the zone?

Anſ. 2136.7712.

3. One end of a circular zone is 48, the other end is 30, and the breadth is 13; what is the area of the zone?[*] *Anſ.* 534.4249.

PROBLEM XV.

To find the area of a circular ring, or the ſpace included between the circumferences of two concentric circles.

---

[*] In any of theſe queſtions, when the radius or diameter of the circle is given, the operation is much more eaſily performed.

RULE.

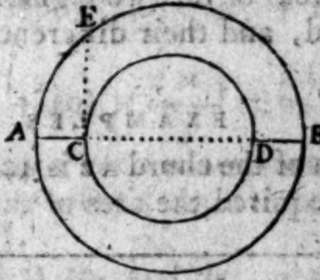
RULE.*

The difference between the areas of the two circles will be the area of the ring.

Or, Multiply the ſum of the diameters by their difference, and this product again by .7854, and it will give the area required.

EXAMPLES.

1. The diameters AB and CD are 20 and 15: required the area of the circular ring, or the ſpace included between the circumferences of thoſe circles.



Here $\overline{AB + CD} \times \overline{AB - CD} = 35 \times 5 = 175$; *and* $175 \times {\cdot}7854 = 137{\cdot}4450$ = *area of the ring required.*

---

* *Demon.* The area of the circle AEBA $= AB^2 \times {\cdot}7854$, and the area of the ſmall circle CD is $= CD^2 \times {\cdot}7854$; therefore the area of the ring $= AB^2 \times {\cdot}7854 - CD^2 \times {\cdot}7854 = \overline{AB + CD} \times \overline{AB - CD} \times {\cdot}7854$. Q. E. D.

Coroll. If CE be a perpendicular at the point C, the area of the ring will be equal to that of a circle whoſe radius is CE.

Rule 2. Multiply half the ſum of the circumferences by half the difference of the diameters, and the product will be the area.

This rule will alſo ſerve for any part of the ring, uſing half the ſum of the intercepted arcs for half the ſum of the circumferences.

2. The diameters of two concentric circles are 16 and 10: what is the area of the ring formed by thoſe circles? *Anſ.* 122.5224.

3. The two diameters are 21.75 and 9.5: required the area of the circular ring. *Anſ.* 300.6609.

4. Required the area of the ring, the diameters of whoſe bounding circles are 6 and 4. *Anſ.* [illegible]708.

PROBLEM XVI.

To find the areas of lunes, or the ſpaces inc[illegible]ed between the interſecting arcs of two eccentric circ[illegible].

RULE.*

Find the areas of the two ſegments from which the lune is formed, and their difference will be the area required.

EXAMPLES.

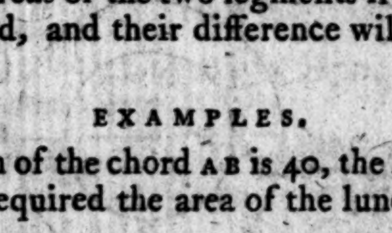
The length of the chord AB is 40, the height DC 10, and DE 4: required the area of the lune ACBEA.

---

* Whoever wiſhes to be acquainted with the properties of lunes, and the various theorems ariſing from them, may conſult *Mr. Whiſton's Commentary on Tacquet's Euclid*, where they will find this ſubject very ingeniouſly managed.

The following property is one of the moſt curious.

If ABC be a right angled triangle, and ſemicircles be deſcribed on the three ſides as diameters, then will the ſaid triangle be equal to the two lunes D and E taken together.



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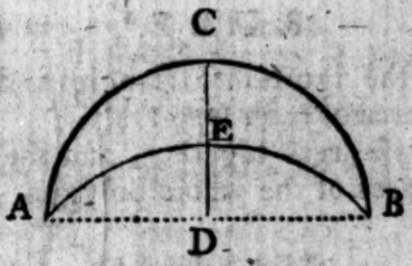


Here $4\sqrt{\overline{AD^2 + DC^2}} = 4\sqrt{\overline{400 + 100}} = \overline{22.36} \times 4$ $= 89.44$.

And $(\frac{89.44}{3} + 40) \times \frac{2 \times 10}{5} = \overline{29.81 + 40} \times 4 =$ $279.24 =$ *area of the ſegment* ABCA.

Alſo $4\sqrt{\overline{AD^2 + DE^2}} = 4\sqrt{\overline{400 + 16}} = 20.396 \times 4$ $= 81.854$.

And $(\frac{81.584}{3} + 40) \times \frac{2 \times 4}{5} = \overline{27.194 + 40} \times 1.6$ $= 107.5104 =$ *area of the ſegment* ABEA.

Whence $279.24 - 107.5104 = 171.7296 =$ *area of the lune required**.

2. The chord is 20, and the heights of the ſegments 10 and 2: required the area of the lune.

Anſ. 128.555.

3. The length of the chord is 48, and the heights of the ſegments 18 and 7: what is the area of the lune? *Anſ.* 405.8676.

PROBLEM XVII.

To find the area of an irregular polygon, or a figure of any number of ſides.

---

* *Note.* The areas of the ſegments in this and the following examples may be more accurately found by Rule 3, page 77.

RULE.*

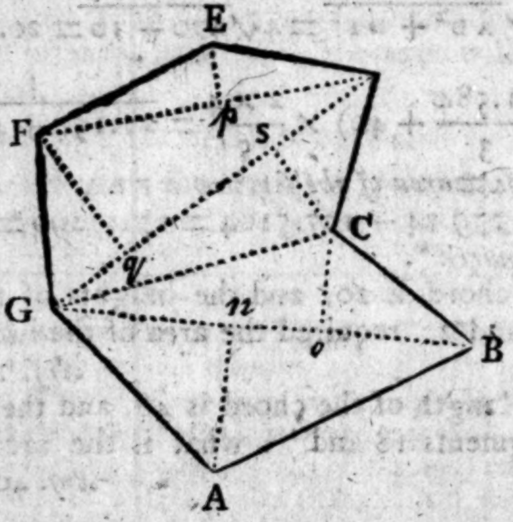
1. Divide the figure into triangles and trapeziums, and find the area of each ſeparately.

2. Add theſe areas together, and the ſum will be equal to the area of the whole polygon,

EXAMPLES.

1. Required the area of the irregular figure ABCDEFGA, the following lines being given.

$GB = 30.5$, $An = 11.2$, $Co = 6$
$GG = 29$ $Fq = 11$ $Cs = 6.6$
$FD = 24.1$ $Ep = 4$ ......



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* When any part of the figure is bounded by a curve, the area may be found as follows:

Rule. 1. Erect any number of perpendiculars upon the baſe, at equal diſtances, and find their lengths.

2. Add the lengths of the perpendiculars, thus found, together, and the ſum divided by their number will give the *mean breadth.*

3. Multiply the mean breadth by the length of the baſe, and it will give the area of that part of the figure required.

Here $\frac{\mathrm{A}n+\mathrm{C}o}{2}+\mathrm{GB}=\frac{11.2+6}{2}\times30.5=8.6\times30.5$ $=262.3=$ *area of the trapezium* ABCG.

And $\frac{\mathrm{F}q+\mathrm{C}s}{2}\times\mathrm{GD}=\frac{11+6.6}{2}\times2.9=8.8\times2.9=$ $255.2=$ *area of the trapezium* GCDF.

Alſo $\frac{\mathrm{FD}\times\mathrm{E}p}{2}=\frac{24.8\times4}{2}=\frac{99.2}{2}=49.6=$ *area of the triangle* FDE.

Whence $262.3+255.2+49.6=567.1=$ *area of the whole figure required.*

2. * In a pentangular field, beginning with the ſouth ſide, and meaſuring round towards the eaſt, the firſt, or ſouth ſide, is 2735 links, the ſecond 3115, the third 2370, the fourth 2925, and the fifth 2220; alſo the diagonal from the firſt angle to the third is 3800 links, and that from the third to the fifth 4010: required the area of the field.

Anſ. 117 *ac.* 2 *ro.* 28 *po.*

To find the area of mixed or compound figures, or ſuch as are compoſed of rectilineal and curvilineal figures together: the rule is to find the area of the ſeveral figures of which the whole figure is compoſed, then add all the areas together, and the ſum will be the area of the whole compound figure. And in the ſame manner may any irregular field or piece of land be meaſured, by dividing it into trapeziums and triangles, and finding the area of each ſeparately.

* *Note.* As this figure conſiſts of three triangles, all of whoſe ſides are given, by calculating their areas according to problem 3, the ſum will be the area of the whole figure accurately, without drawing perpendiculars from the angles to the diagonals.

The ſame thing may alſo be done in moſt other caſes of this kind.

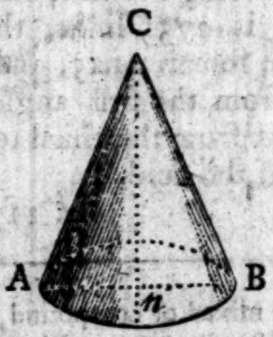
OF THE

CONIC SECTIONS.

DEFINITIONS.

1. THE *conic ſections* are ſuch plain figures as are formed by the cutting of a cone.

2. * A *cone* is a ſolid deſcribed by the revolution of a right-angled triangle about one of its legs, which remains fixed.



3. The *axis of a cone* is the right line about which the triangle revolves.

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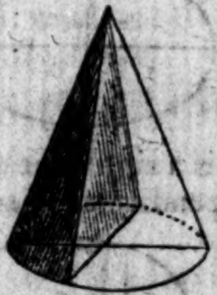
* This is Euclid's definition of a cone, and is that which is generally beſt underſtood by a learner; but the following one is more general.

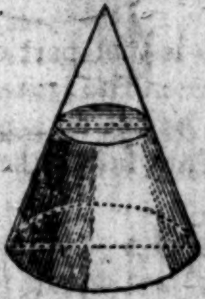
Conceive the right line C B to move upon the fixed point C as a centre, and ſo as continually to touch the circumference of the circle A B, placed in any poſition, except in that of a plane which paſſes through the ſaid point; and then that part of the line which is intercepted between the fixed point and the periphery of the circle will generate the convex ſuperficies of a cone.

4. The *baſe of a cone* is the circle which is deſcribed by the revolving leg of the triangle.

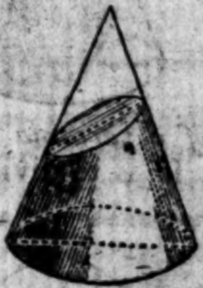
5. If a cone be cut through the vertex, by a plane perpendicular to that of the baſe, the ſection will be a *triangle*.



6. If a cone be cut into two parts, by a plane parallel to the baſe, the ſection will be a *circle*.



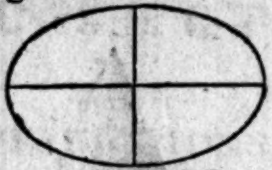
7. If a cone be cut by a plane which paſſes through its two ſlant ſides in an oblique direction, the ſection will be an *ellipſis*.



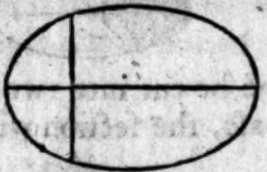
8. If two lines be drawn through the centre of an ellipſis, perpendicular to each other, and terminated

both ways by the circumference, they are called the *tranſverſe* and *conjugate diameters*, or *axes*.

The longeſt line is the tranſverſe axis, and the ſhorteſt the conjugate.

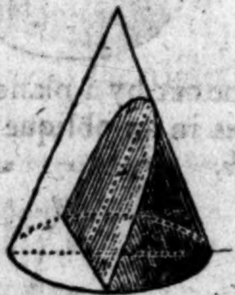


9. *An ordinate* is a right line drawn from any point of the curve, perpendicular to either of the diameters.

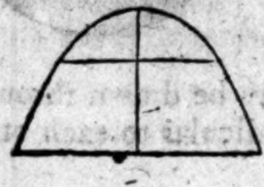


10. *An abſciſſa* is that part of the diameter which is contained between the vertex and the ordinate.

11. If a cone be cut by a plane, which is parallel to either of its ſlant ſides, the ſection will be *a parabola*.



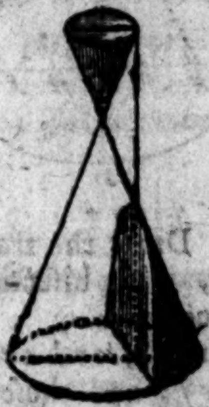
12. *The axis of a parabola* is a right line drawn from the vertex, ſo as to divide the figure into two equal parts.



13. *The ordinate* is a right line drawn from any point in the curve perpendicular to the axis.

14. *The abſciſſa* is that part of the axis which is contained between the vertex and the ordinate.

15. * If a cone be cut into two parts, by a plane, which being continued, would meet the oppoſite cone, the ſection is called an *hyperbola*.



16. *The tranſverſe diameter* or axis of an hyberbola is that part of the axis intercepted between the two oppoſite cones.

17. *The conjugate diameter* is a line drawn through the centre perpendicular to the tranſverſe.

18. *An ordinate* is a line drawn from any point in the curve perpendicular to the axis; and the *abſciſſa* is the diſtance intercepted between that ordinate and the vertex.

---

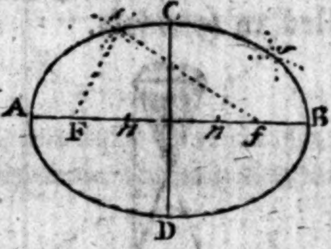
* The two oppoſite cones, in this definition, are ſuppoſed to be generated together, by the revolution of the ſame line.

All the figures which can poſſibly be formed by the cutting of a cone, are mentioned in theſe definitions, and are the five following ones; *viz.* a *triangle*, a *circle*, an *ellipſis*, a *parabola*, and an *hyperbola*; but the three laſt only are uſually called the *conic ſections*.

PRO-

PROBLEM I.

To deſcribe an ellipſis, the tranſverſe and conjugate diameters being given.



*Conſtruction.** 1. Draw the tranſverſe and conjugate diameters, A B, C D, biſecting each other perpendicularly in the centre *o*.

2. With the radius A *o*, and centre C, deſcribe an arc, cutting A B in F *f*; and theſe two points will be the foci of the ellipſe.

3. Take any number of points *n n*, &c. in the tranſverſe diameter A B, and with the radii A *n*, *n* B,

---

* It is a known property of the ellipſe, that the ſum of two lines drawn from the foci, to meet in any point in the curve, is equal to the tranſverſe diameter, and from this the truth of the conſtruction is evident.

From the ſame principle is alſo derived the following method of deſcribing an ellipſe, by means of a ſtring and two pins.

Having found the foci F, *f*, as before, take a thread of the length of the tranſverſe diameter, and faſten its ends with two pins in the points F, *f*; then ſtretch the thread F *s f* to its greateſt extent, and it will reach to the point *s* in the curve; and by moving a pencil round within the thread, keeping it always ſtretched, it will trace out the curve required.

and centres F *f* deſcribe arcs interſecting each other in *s*, *s*, &c.

4. Through the points *s*, *s*, &c. draw the curve A*s*CBD, and it will be the circumference of the ellipſe required.

PROBLEM II.

In an ellipſis, any three of the four following terms being given, viz. the tranſverſe and conjugate diameters, an ordinate and its abſciſſa, to find the fourth.

CASE I.

When the tranſverſe, conjugate, and abſciſſa are known, to find the ordinate.

RULE.*

As the tranſverſe diameter is to the conjugate,
So is the ſquare root of the rectangle of the two abſciſſas,
To the ordinate which divides them.

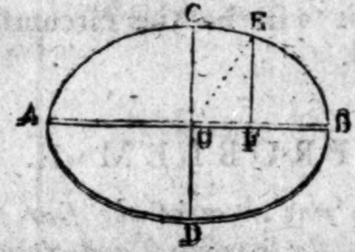
EXAMPLES.

1. In the ellipſis ADBC, the tranſverſe diameter AB is 120, the conjugate diameter CD is 40, and

---

* Let $t =$ the tranſverſe diameter, $c =$ conjugate, $x =$ any abſciſſa, and $y =$ ordinate. Then will the general equation, expreſſing the property of the ellipſe, be $t^2 : c^2 :: x \times (t-x) : y^2$; and from this the four rules here given are deduced, the one above being $y = \frac{c}{t} \sqrt{x \times \overline{t-x}}$.

the abſciſſa BF 24: what is the length of the ordinate EF?



Here 120 (AB) : 40 (CD) :: $\sqrt{96\times24}$ (AF×FB) : $\frac{40}{120}\sqrt{96\times24}=\frac{1}{3}\sqrt{2304}=\frac{1}{3}\times48=16=$ EF *the ordinate required.*

2. If the tranſverſe diameter be 35, the conjugate 25, and the abſciſſa 28: what is the ordinate?

Anſ. 10.

CASE II.

When the tranſverſe, conjugate, and ordinate are known, to find the abſciſſa.

RULE.*

As the conjugate diameter is to the tranſverſe,
So is the ſquare root of the difference of the ſquares of the ordinate and ſemi-conjugate,
To the diſtance between the ordinate and centre.
And this diſtance being added to and ſubtracted from the ſemi-tranſverſe, will give the two abſciſſas required.

---

[*] This rule in algebraic terms is as follows: The greater abſciſſa $x=t+\frac{t}{c}\sqrt{\frac{1}{4}c^2-y^2}$, or the leſs $x=t-\frac{t}{c}\sqrt{\frac{1}{4}c^2-y^2}$.

EXAMPLES.

1. The tranſverſe diameter AB is 120, the conjugate diameter CD is 40, and the ordinate FE is 16: what is the abſciſſa FB?

Here $40(\mathrm{CD}):120(\mathrm{AB})::\sqrt{20^2-16^2}\,(\sqrt{\mathrm{OB}^2-\mathrm{FE}^2})$ $:\frac{120}{40}\sqrt{20^2-16^2}=3\sqrt{400-256}=3\sqrt{144}=3\times12=36=\mathrm{OF}$, *the diſtance from the centre.*

$$\left.\begin{array}{l}\textit{Whence } 60\,(\mathrm{OB})-36\,(\mathrm{OF})=24=\mathrm{BF}\\ \textit{And } 60\,(\mathrm{OA})+36\,(\mathrm{OF})=96=\mathrm{AF}\end{array}\right\}=\textit{two}$$

abſciſſas required.

2. What are the two abſciſſas to the ordinate 10, the diameters being 35 and 23?

Anſ. 7 *and* 28.

CASE III.

When the conjugate, ordinate, and abſciſſa are known, to find the tranſverſe.

RULE.*

1. To, or from, the ſemi-conjugate, according as the leſs or great abſciſſa is uſed, add or ſubtract the ſquare root of the difference of the ſquares of the ordinate and ſemi-conjugate.

2. Then, as the ſquare of the ordinate, is to the rectangle of the conjugate and abſciſſa, ſo is the ſum or difference above found, to the tranſverſe diameter required.

---

* This rule in algebraic terms is as follows: $t=(c+2\sqrt{\tfrac{1}{4}c^2-y^2})\times\frac{cx}{2y^2}$, or $t=(c-2\sqrt{\tfrac{1}{4}c^2-y^2})+\frac{cx}{2y^2}$ according as the greater or leſs abſciſſa is uſed.

EXAMPLES.

1. The conjugate diameter CD is 40, the ordinate EF is 16, and the abſciſſa FB 24: required the tranſverſe AB.

Here $20\ (OB) + \sqrt{\overline{20 - 16^2}}\ (\sqrt{\overline{OB^2 - EF^2}}) = 20 + \sqrt{\overline{400 - 256}} = 20 + \sqrt{144} = 20 + 12 = 32$.

And $16^2\ (EF^2) : 40 \times 24\ (CD \times BF) :: 32 : \frac{40 \times 24 \times 32}{16^2} = \frac{40 \times 24 \times 2}{16} = \frac{5 \times 24 \times 2}{2} = 5 \times 24 =$ 120 *the tranſverſe diameter required.*

2. If an ordinate and its leſſer abſciſſa be 10 and 7, and the conjugate 25: what is the tranſverſe?

Anſ. 35.

CASE IV.

The tranſverſe, ordinate, and abſciſſa being given, to find the conjugate.

RULE.*

As the ſquare root of the product of the two abſciſſas, is to the ordinate,
So is the tranſverſe diameter to the conjugate.

EXAMPLES.

1. The tranſverſe AB is 120, the ordinate EF 16, and the abſciſſa FB 24: required the conjugate.

---

[*] The rule in algebraic terms is $ty \times \frac{1}{\sqrt{\overline{t - x.x}}} = c$, the conjugate, or ſhorteſt diameter.

Here $\sqrt{\overline{24 \times 96}}\ (\sqrt{\overline{\text{BF} \times \text{AF}}}) : 16\ (\text{FF}) :: 120\ (\text{AB}) : \overline{16 \times 120} \div \sqrt{\overline{24 \times 96}} = \overline{16 \times 120} \div \sqrt{2304} = \overline{16 \times 120} \div 48 = \frac{15 \times 120}{48} = \frac{120}{3} = 40$ *the conjugate diameter required.*

2. The tranſverſe diameter is 35, the ordinate is 10, and its abſciſſa 7: what is the conjugate?

Anſ. 25.

PROBLEM III.

To find the circumference of an ellipſe, the tranſverſe and conjugate diameters being known.

RULE.*

Multiply the ſquare root of half the ſum of the ſquares of the two diameters by 3.1416, and the product will be the circumference *nearly*.

EXAM-

---

* *Demon.* Let $t =$ tranſverſe diameter, $c =$ conjugate, $p = 3.1416$, and $d = 1 - \frac{c^2}{t^2}$. Then will $pt \times (1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$ &c.) $=$ circumference of the ellipſe, as is ſhewn by the writers on fluxions.

Now the rule given above is $p \sqrt{\frac{t^2 + c^2}{2}} = pt \sqrt{(\frac{1}{2} + \frac{c^2}{2t^2})} = pt \sqrt{(1 - \frac{1}{2} + \frac{c^2}{2t^2})} =$ (by ſubſtitution) $pt \sqrt{(1 - \frac{d}{2})} = pt \times : 1 - \frac{d}{2.2} - \frac{d^2}{2^3.4} - \frac{3d^3}{2^4.4.6}$ &c. But the three firſt

terms

EXAMPLES.

1. The tranſverſe diameter is 24 and the conjugate 20: required the circumference of the ellipſe.

Here $\sqrt{\frac{AB^2 - CD^2}{2}} = \sqrt{\frac{24^2 + 20^2}{2}} = \sqrt{\frac{576 + 400}{2}}$ $= \sqrt{288 + 200} = 488 = 22.09.$

And 22.09 × 3.1416 = 69.397944 *the circumference required.*

2. The axes are 24 and 18: what is the circumference? *Anſ.* 66.6433.

---

terms of this ſeries differ from the three firſt terms of the former only by $\frac{d^2}{64}$; therefore the rule is ſhewn to be an approximation. Q. E. D.

Rule 2. Multiply ½ the ſum of the two diameters by 3.1416, and the product will give the circumference *exact enough to anſwer moſt practical purpoſes.*

Rule 3. Find the circumference both from the laſt rule and that given above, and ½ the ſum of the reſults will give the circumference *extremely near.*

Note. If a = ſemi-tranſverſe B O, c = ſemi-conjugate C O, and x = diſtance O F, of the ordinate E F from the centre, then will the arc C E be $= x \times (1 \pm \frac{c^2}{6a^4} x^2 + \frac{4a^2 c^2 - c^4}{4ca^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}} x^6$ &c.

The following may ſerve as a practical rule for finding the length of the arc C E.

Find the length of a circular arc intercepted by O E and O C, and whoſe radius is ½ the ſum of O E and O C, and it will be the elliptic arc C E *nearly*.

PRO-

PROBLEM IV.

To find the area of an ellipſe, the tranſverſe and conjugate diameters being given.

RULE.*

Multiply the tranſverſe diameter by the conjugate, and the product again by .7854, and the reſult will be the area.

Or multiply .7854 by one axe, and the product by the other.

EXAMPLES.

1. What is the area of an ellipſe whoſe tranſverſe diameter is 24, and the conjugate 18?

Here 24×18×.7854=339.2928=*area required.*

Or,

.7854
24=*tranſverſe.*

31416
15708

18.8496
18=*conjugate.*

1507968
188496

339.2928=*area before.*

2. If the axes of an ellipſe be 35 and 25, what is the area? *Anſ.* 687.225.

3. Required the area of an ellipſis whoſe two axes are 70 and 50. *Anſ.* 2748.9.

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* The demonſtration of this rule is contained in that of the next problem.

PROBLEM V.

To find the area of an elliptic ſegment, whoſe baſe is parallel to either of the axes of the ellipſe.

RULE.[*]

1. Divide the height of the ſegment by that axe of the ellipſe of which it is a part, and find, in the table, a circular ſegment whoſe verſed ſine is equal to the quotient.

2. Multiply the ſegment thus found, and the two axes of the ellipſe continually together, and the product will give the area required.

EXAMPLES.

1. Required the area of the elliptic ſegment EAF, whoſe height AG is 10, and the axes of the ellipſe 2AB and CD, 35 and 25 reſpectively.

10.0000

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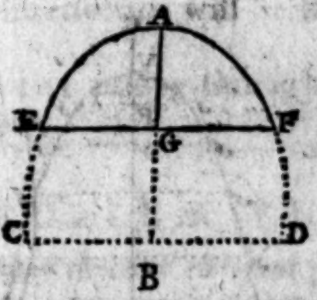
[*] *Demon.* Let the tranſverſe diameter 2AB $= t$, the conjugate CD $= c$, AG $= x$, and EG $= y$; then by the property of the curve we ſhall have $y = \frac{c}{t}\sqrt{tx - x^2}$, and the fluxion of the area EAF $= (y\dot{x}) = \frac{c}{t} \times \dot{x}\sqrt{tx - x^2}$. But $\dot{x} \times \sqrt{tx - x^2}$ is known to expreſs the fluxion of the correſponding circular ſegment, whoſe verſed ſine is x, and the diameter t. Let the fluent of this expreſſion therefore be denoted by A, and then the fluent of $\frac{c}{t} \times \dot{x}\sqrt{tx - x^2}$ will be $= \frac{c}{t} \times$ A, from whence the rule is derived. Q. E. I.

Coroll. The ellipſe is equal to a circle whoſe diameter is a mean proportional between the two axes, and from hence the rule is formed for the whole ellipſe.



Here $\frac{10.0000}{35}=\frac{2.0000}{7}=.2857=$ *tabular verſed ſine.*

And the tabular ſegment belonging to this is .185153.

Whence $.185153\times35(2AB)\times25(CD)=6.480355\times25=162.0210=$ *area of the ſegment required.*

2. What is the area of an elliptic ſegment cut off by a double ordinate parallel to the conjugate axe, at the diſtance of 36 from the centre, the axes being 120 and 40? *Anſ.* 536.7504.

3. What is the area of a ſegment, cut off by an ordinate parallel to the tranſverſe diameter, whoſe height is 5, the axes being 35 and 25? *Anſ.* 97.845725.

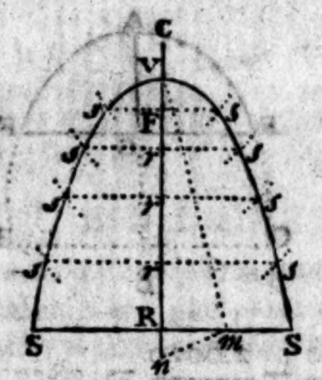
PROBLEM VI.

To deſcribe a parabola, any ordinate to the axe and its abſciſſa being given.

The area of an elliptic ſegment may alſo be found by the following rule.

Find the correſponding ſegment of the circle deſcribed upon the ſame axe to which the baſe of the ſegment is perpendicular.

Then as this axe is to the other axe, ſo is the circular ſegment to the elliptic ſegment.



* *Conſtruction.* 1. Biſect the given ordinate RS in m; join Vm, and draw mn perpendicular to it, meeting the axe in n.

2. Make VC and VF each equal to Rn, and F will be the focus of the curve.

3. Take any number of points r, r, &c. in the axe, through which draw the double ordinates srs, srs, &c. of an indefinite length.

4. With the radii CF, Cr, &c. and centre F, deſcribe arcs cutting the correſponding ordinates in the points s, s, &c. and the curve sVs drawn through all.

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* Since Vmn is a right angled triangle, and mR is perpendicular to Vn, $VR \times Rn = VR \times VF = Rm^2 = \frac{1}{4} Rs^2$, which is a known property of the parabola when F is the focus. And, becauſe $sF^2 = cr^2 - rF^2 = \overline{cr + rF} \times \overline{cr - rF} = \overline{cr + rF} \times cF = 2Vr \times 2VF = Vr \times 4VF$, therefore s is a point in the curve of a parabola, and the ſame may be ſhewn of any other point s.

Beſides the methods above, for finding the focus, it may be found arithmetically as follows:

Divide the ſquare of the ordinate by 4 times the abſciſſa, and the quotient will be the focal diſtance vF.

Several other methods of doing this, as well as of deſcribing the curve itſelf, may alſo be found in Emerſon's Conic Sections, and other performances.

all the points of interſection will be the parabola required.

Note. The line ſ F ſ paſſing through the focus F is called the parameter.

PROBLEM VII.

In a parabola, any three of the four following terms being given, viz. any two ordinates and their two abſciſſas, to find the fourth.

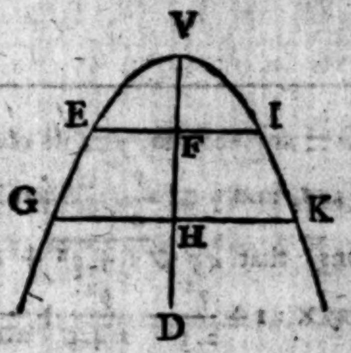
RULE.*

As any abſciſſa is to the ſquare of its ordinate, ſo is any other abſciſſa to the ſquare of its ordinate.

Or as the ſquare root of any abſciſſa is to its ordinate, ſo is the ſquare root of any other abſciſſa to its ordinate.

EXAMPLES.

1. The abſciſſa V F is 9, and its ordinate E F 6; required the ordinate G H, whoſe abſciſs V H is 16.



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* If x and X be any two abſciſſas, and y and Y their correſponding ordinates, the equation of the curve will be $x\mathrm{Y}^2 = \text{Cap.}\ \mathrm{X}.\ y^2$, which is the ſame as the rule.

Here $\sqrt{9}$ ($\sqrt{\text{VF}}$) : 6 (EF) :: $\sqrt{16}$ ($\sqrt{\text{VH}}$) : $\frac{6\times\sqrt{16}}{\sqrt{9}}=\frac{6\times4}{3}=\frac{24}{3}=8=$ *ordinate* GH.

Or,

9 (VF) : 36 (EF^2) :: 16 (VH) : $\frac{16\times36}{9}=16\times4=64=\text{GH}^2$, or $8=\text{GH}$ *as before.*

2. The two abſciſſas are 9 and 16, and their correſponding ordinates 6 and 8; from any three of theſe to find the fourth.

PROBLEM VIII.

To find the length of any arc of a parabola, cut off by a double ordinate.

RULE.*

To the ſquare of the ordinate add $\frac{4}{3}$ of the ſquare of the abſciſſa, and twice the ſquare root of the ſum will be the length of the curve required.

EXAM-

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* *Demon.* Let $x=$ any abſciſſa, $y=$ its ordinate, $a=\frac{1}{2}$ the parameter of the axe, and $q=\frac{y}{a}$. Then it is ſhewn by the writers on fluxions, that $aq\sqrt{1+q^2}+a\times$ hyp. log. of $(q+\sqrt{1+q})=2y\times:1+\frac{q^2}{2.3}-\frac{q^4}{2.4.5}+\frac{3q^6}{2.4.6.7}-\frac{3.5q^8}{2.4.6.8.9}$ &c. $=c=$ length of the curve. But $\sqrt{1+\frac{1}{3}q^2}=1+\frac{q^2}{2.3}-\frac{q^4}{2.4.9}+\frac{3q^6}{2.4.6.27}$ &c. agreeing with the former in the two firſt terms.

Therefore

EXAMPLES.

1. The abſciſſa VH is 2, and its ordinate GH 6: what is the length of the arc GVK?

Here 2^2 (VH^2) $\times \frac{4}{3} + 36$ (GH^2) $= \frac{4\times 4}{3} + 36 = 5\frac{1}{3} + 36 = 5.333$, &c. $+ 36 = 41.333333$.

```
      .  .  .  .  .
And 41.3333333(6.429
    36                2
    ——            —————
124)533           12.858 = length of the arc.
    496
    ———
1282)3733
     2564
     ————
12849)116933
      115941
      ——————
        1292
```

PROBLEM IX.

To find the area of a parabola, its baſe and height being given.

RULE.*

Multiply the baſe by the height, and $\frac{2}{3}$ of the product will be the area required.

EXAM-

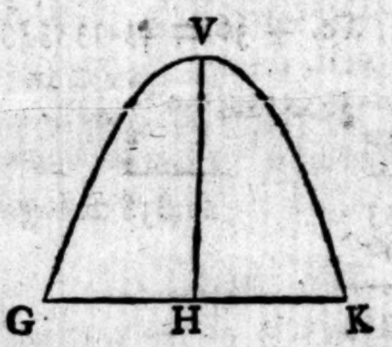
Therefore $\frac{c}{2y} = \sqrt{1 + \frac{1}{3}q^2}$ *nearly*; and conſequently $c = 2y\sqrt{1 + \frac{1}{3}q^2} = \sqrt{y^2 + \frac{4}{3}x^2}$ the ſame as the rule. Q. E. D.

* *Demon.* Let VH $= x$, GH $= y$, and the parameter $= p$. Then $pz = y^2$ or $\sqrt{px} = y$ by the nature of the curve.

Whence

EXAMPLES.

1. What is the area of the parabola GVK, whoſe height VH is 12, and the baſe or double ordinate GK 16?



Here $16\,(GK) \times 12\,(VH) \times \frac{2}{3} = \frac{16 \times 12 \times 2}{3} =$ $16 \times 4 \times 2 = 128 =$ *area required.*

2. The abſciſſa is 12, and the double ordinate or baſe 38: what is the area? *Anſ.* 304.

3. What is the area of a parabola whoſe abſciſſa is 10, and ordinate 8? *Anſ.* $106\frac{2}{3}$.

PROBLEM X.

To find the area of the fruſtum of a parabola.

Whence the fluxion of the area ($= y\dot{x}$) $= \dot{x}\sqrt{px}$, and its fluent $= \frac{2}{3}x \times \sqrt{px}$.

But becauſe $y = \sqrt{px}$, therefore $\frac{2}{3}x \times \sqrt{px} = \frac{2}{3}xy =$ area of the parabola, which is the ſame as the rule.

Coroll. Every parabola is $= \frac{2}{3}$ of its circumſcribing parallelogram.

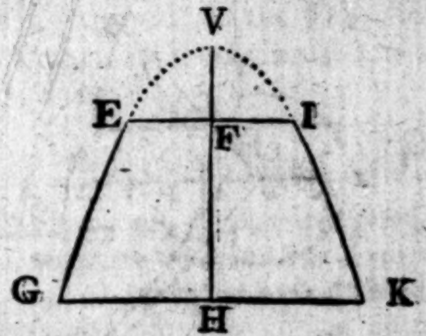
RULE.

RULE.*

Divide the difference of the cubes of the two ends of the fruſtum by the difference of their ſquares, and this quotient multiplied by $\frac{2}{3}$ of the altitude, will give the area required.

EXAMPLES.

1. In the parabolic fruſtum GEIK, the two parallel ends EI and GK are 6 and 10, and the altitude, or part of the abſciſſa FH, is 3: what is the area?



Here $\overline{10^3-6^3}$ (GK^3-EI^3) $\div$ $\overline{10^2-6^2}$ (GK^2-EI^2)

$$=\frac{10^3-6^3}{10^2-6^2}=\frac{1000-216}{100-36}=\frac{784}{64}=\frac{98}{8}=\frac{49}{4}=12.25.$$

$$\textit{And } 12.25\times\frac{2\times 3}{3}=12.25\times 2=24.5=\textit{area required.}$$

---

* *Demon.* Let $D=GK$, $d=EI$, and $a=FH$.

Then by the nature of the curve $D^2-d^2 : a :: D^2 : \frac{aD^2}{D^2-d^2}=VH$, and $D^2-d^2 : a :: d^2 : \frac{ad^2}{D^2-d^2}=VF$.

And therefore $\frac{2}{3}\times\frac{aD^3}{D^2-d}-\frac{2}{3}\times\frac{ad^3}{D^2-d^2}=\frac{2}{3}a\times\frac{D^3-d^3}{D^2-d^2}=$ area of the fruſtum. Q.E.D.

2. The greater end of the fruſtum is 24, the leſſer end is 20, and their diſtance $5\frac{1}{2}$: what is the area?

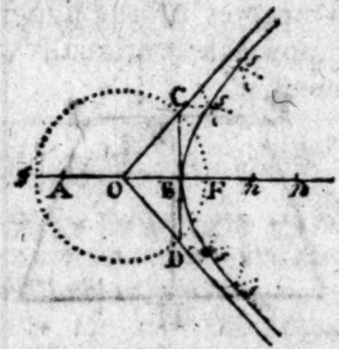
Anſ. 121.3.

3. Required the area of the parabolic fruſtum, the greater end of which is 10, the leſs 6, and the height 4.

Anſ. $32\frac{2}{3}$.

PROBLEM XI.

To conſtruct an hyperbola, the tranſverſe and conjugate diameters being given.



* *Conſtruction.* 1. Make A B the tranſverſe diameter, and C D perpendicular to it, the conjugate.

Note. Any parabolic fruſtum, is equal to a parabola of the ſame altitude, whoſe baſe is equal to one end of the fruſtum, increaſed by a third proportional to the ſum of the two ends and the other end.

* The ſum of two lines drawn from the foci of an ellipſe to any point in the curve, is equal to its tranſverſe diameter.

In like manner the difference of two lines drawn from the foci of any hyperbola to any point in the curve, is equal to its tranſverſe diameter; as is ſhewn by the writers on *conics*.

But the arcs interſecting each other in *s*, *s*, &c. are deſcribed from the foci *f* and F, and with the diſtances A *n* and B *n*, whoſe difference is A B, and therefore the points *s*, *s*, &c. are in the curve of an hyperbola.

2. Biſect AB in O, and from O with the radius OC, or OD, deſcribe the circle DfCF, cutting AB produced in F and f, which points will be the two foci.

3. In AB produced take any number of points, n, n, &c. and from F and f, as centers, with the diſtances Bn, An, as radii, deſcribe arcs cutting each other in s, s, &c.

4. Through the ſeveral points s, s, &c. draw the curve sBs, and it will be the hyperbola required.

5. If ſtraight lines be drawn from the point O, through the extremities CD of the conjugate axis, they will be the aſymptotes of the hyperbola, whoſe property it is to approach continually to the curve, and yet never to meet it.

PROBLEM XII.

In an hyperbola, any three of the four following terms being given, viz. the tranſverſe and conjugate diameters, an ordinate and its abſciſſa, to find the fourth.

CASE I.

The tranſverſe and conjugate diameters, and the two abſciſſas being known, to find the ordinate.

RULE.*

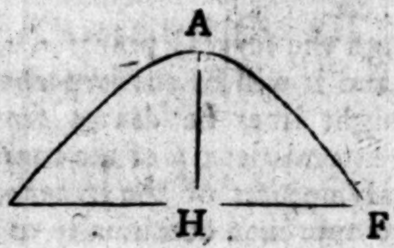
As the tranſverſe diameter,
Is to the conjugate;
So is the ſquare root of the product of the two abſciſſas,
To the ordinate required.

---

* Let t = tranſverſe diameter, c = conjugate, x = abſciſſa, and y = ordinate. Then the general property of the curve is $t^2 : c^2 :: x \times (t+x) : y^2$; and from this analogy all the caſes of this problem are deduced.

EXAMPLES.

1. In the hyperbola GAF, the tranſverſe diameter is 120, the conjugate 72, and the abſciſſa AH is 40: required the ordinate FH.



$$120\ (\textit{tranſ.}) : 72\ (\textit{conj.}) :: \surd(160\times40) : \frac{72\times\surd(160\times40)}{120} = \frac{6\times\surd(160\times40)}{10} = \tfrac{2}{5}\surd(160\times40) = \tfrac{3}{5}\surd6400 = \tfrac{3}{5}\times80 = \frac{3\times80}{5} = 3\times16 = 48 = \textit{ordinate}\ FH.$$

2. The tranſverſe diameter is 24, the conjugate 21, and the leſſer abſciſſa 8: what is its ordinate?

Anſ. 14.

3. The tranſverſe diameter of an hyberbola is 50, the conjugate 30, and the greater abſciſſa 12: required the ordinate.

CASE II.

The tranſverſe and conjugate diameters, and an ordinate being given, to find the two abſciſſas.

Note. In the hyperbola, the leſs abſciſſa added to the axis, gives the greater.

RULE.

As the conjugate diameter is to the tranſverſe,
So is the ſquare root of the ſum of the ſquares of the ordinate and ſemi-conjugate,
To the diſtance between the ordinate and the centre, or half the ſum of the abſciſſas.

Then the ſum of this diſtance and the ſemi-tranſverſe will give the greater abſciſſa, and their difference the leſſer abſciſſa required*.

EXAMPLES.

The tranſverſe diameter is 120, the conjugate 72, and the ordinate 48: what are the two abſciſſas?

1296 = *ſquare of the ſemi-conjugate.*
2304 = *ſquare of the ordinate.*

3600(60 = *ſquare root.*
36

00

As 72 : 120 : : 60
120

72)7200(100 = ½ *ſum of the abſciſſas.*
72   60 = *ſemi-tranſverſe.*

160 = *greater abſciſſa.*

40 = *leſſer abſciſſa.*

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* This rule in ſpecies is $\frac{t}{c}\sqrt{\frac{1}{4}c^2+y^2}\pm\frac{1}{2}t=x$, = greater or leſs abſciſſa, according as the upper or under ſign is uſed.

2. The tranſverſe and conjugate diameters are 24 and 21: required the two abſciſſas to the ordinate 14.

Anſ. 32 *and* 8.

3. The tranſverſe being 60, and the conjugate 36; required the two abſciſſas to the ordinate 24.

Anſ. 80 *and* 20.

CASE III.

The tranſverſe diameter, the two abſciſſas and the ordinate being given, to find the conjugate.

RULE.

As the ſquare root of the product of the two abſciſſas,
Is to the ordinate;
So is the tranſverſe diameter,
To the conjugate*.

EXAMPLES.

1. The tranſverſe diameter is 120, the ordinate is 48, and the two abſciſſas are 160 and 40: required the conjugate.

$$\begin{array}{l} 160 \\ 40 \\ \hline \dot{6}4\dot{0}0(80 \\ 64 \\ \hline 00 \end{array}$$

---

* This rule expreſſed algebraically is $ty \div \sqrt{x \times (t+x) \times x} = c =$ conjugate diameter.

As 80 : 48 :: 120 *the tranſverſe axis.*

$$\begin{array}{r} 120 \\ 48 \\ \hline 960 \\ 480 \\ \hline 8.0)576.0 \\ \hline 72 \end{array}$$

72=*conjugate required.*

2. The tranſverſe diameter is 24, the ordinate 14, and the abſciſſa 8 and 32: required the conjugate.

Anſ. 21.

CASE IV.

The conjugate diameter, the ordinate and two abſciſſas being given, to find the tranſverſe.

RULE.

1. Add the ſquare of the ordinate to the ſquare of the ſemi-conjugate, and find the ſquare root of their ſum.

2. Take the ſum or difference of the ſemi-conjugate and this root, according as the leſs or greater abſciſſa is uſed, and then ſay,

As the ſquare of the ordinate,
Is to the product of the abſciſſa and conjugate;
So is the ſum, or difference, above found,
To the tranſverſe required[*].

---

[*] This rule in algebraic terms is $\frac{cx}{y^2} \times (\sqrt{\frac{1}{4}c^2 + y^2} = \frac{1}{2}c) = t$ = tranſverſe diameter.

EXAMPLES.

1. The conjugate diameter is 72, the ordinate is 48, and the leſſer abſciſſa 40: what is the tranſverſe?

Here $\sqrt{48^2+36^2}=\sqrt{2304+1296}=\sqrt{3600}=60$, *and* $60+36=96$.

Alſo $72\times40=2880=$*product of the abſciſſa and conjugate. Whence,*

As 2304 : 2880 :: 96
96

17280
25920

2304)276480(120=*tranſverſe required.*
2304

4608
4608

2. The conjugate diameter is 21, the leſſer abſciſſa 8, and its ordinate 14: required the tranſverſe.

Anſ. 24.

3. Required the tranſverſe diameter of the hyperbola, whoſe conjugate is 36, the leſſer abſciſſa being 20. *Anſ.* 60.

PROBLEM XIII.

To find the length of any arc of an hyperbola, beginning at the vertex.

RULE.

RULE.*

1. To 19 times the tranſverſe, add 21 times the parameter of the axis, and multiply the ſum by the quotient of the abſciſſa divided by the tranſverſe.

2. To 9 times the tranſverſe, add 21 times the parameter, and multiply the ſum by the quotient of the abſciſſa divided by the tranſverſe.

3. To each of the products, thus found, add 15 times the parameter, and divide the former by the latter; then this quotient being multiplied by the ordinate will give the length of the arc *nearly*.

EXAMPLES.

1. In the hyperbola GAB, the tranſverſe diameter is 80, the conjugate 60, the ordinate GH 10, and the abſciſſa AH 2.1637: required the length of the arc AG.

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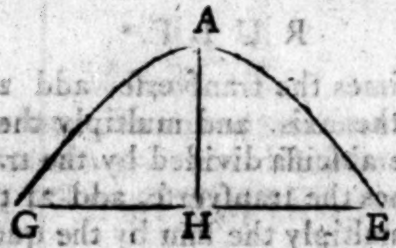
* *Demon.* Let $t =$ ſemi-tranſverſe axe, $c =$ ſemi-conjugate, $x =$ ordinate, and $y =$ abſciſſa. Then will $y \times (1 + \frac{t^2}{6c^4} y^2 + \frac{t^4 + 4t^2 c^2}{40c^8} y^4 + \frac{t^6 + 4t^4c^2 + 8t^2c^4}{112c^{12}} y^6$, &c.$) =$ length of the arc, as is ſhewn by the writers on fluxions.

But $x = \frac{t}{c} \sqrt{c^2 + y^2} - a$, and $\frac{2c}{t} =$ parameter $= p$, by the nature of the curve. Conſequently the rule becomes $(15p + \overline{19t + 21p} \times \frac{x}{t}) \div (15p + \overline{9t + 21p} \times \frac{x}{t}) \times y = (15pt + 19tx + 21px) \div (15pt + 9tx + 21px) \times y = y \times : 1 + \frac{2x}{3p} - \frac{2x^2}{5p}$ &c. which by ſubſtituting the values of x and p, and expanding the terms, gives a ſeries, agreeing nearly in the three firſt terms with the former, and therefore the rule is an approximation.



Here $80 : 60 : 60 : \frac{60 \times 60}{80} = \frac{3 \times 60}{4} = 3 \times 15 = 45$ $=$ *parameter.*

And $\overline{80 \times 19 \times 45 \times 21} \times \frac{2.1637}{80} = \overline{1520 + 945} \times .02704 = 2465 \times .02704 = 66.6536$.

Alſo $\overline{80 \times 9 + 45 \times 21} \times \frac{2.1637}{83} = \overline{720 + 645} \times .0274 = 1665 \times .02704 = 45.0216$.

Whence $\overline{675 + 66.6536} \div \overline{675 \times 45.0216} = 741.6536 \div 720.0216 = 1.03004$; *and* $1.03004 \times 10 = 10.3004$ $=$ *length of the arc required.*

2. The tranſverſe diameter of an hyperbola is 120, the conjugate 72, the ordinate 48, and the abſciſſa 40: required the length of the curve.

Ans. 62.6496.

3. Required the whole length of the curve of an hyperbola, to the ordinate 10; the tranſverſe and conjugate axes being 10 and 60. *Anſ.* 20.601.

PROBLEM XIV.

To find the area of an hyperbola, the tranſverſe, conjugate and abſciſſa being given.

If $t =$ ſemi-tranſverſe, $c =$ ſemi-conjugate, and $y =$ ordinate drawn from the end of the required arc; then $y \times (1 + \frac{t^2y^2}{6c^4} \text{A} - \frac{t^2 + 4c^2}{c^2} . \frac{3y^2}{20} \text{B} + \frac{t^4 + 4t^2c^2 + 8c^4}{t^2 + 4c^2} . \frac{5y^2}{14c^4} \text{C}$ &c.) $=$ length of the arc.

RULE.

RULE.*

1. To the product of the tranſverſe and abſciſſa, add $\frac{5}{7}$ of the ſquare of the abſciſſa, and multiply the ſquare root of the ſum by 21.

2. Add 4 times the ſquare root of the product of the tranſverſe and abſciſſa, to the product laſt found, and divide the ſum by 75.

3. Divide 4 times the product of the conjugate and abſciſſa by the tranſverſe, and this laſt quotient multiplied by the former will give the area required *nearly*.

EXAMPLES.

In the hyperbola GAB, the tranſverſe axis is 30, the conjugate 18, and the abſciſſa or height AH is 10: what is the area?

Here

---

* *Demon.* Let t = tranſverſe diameter, c = conjugate, x = abſciſſa, y = ordinate, and $z = \frac{x}{t2+x}$. Then it is well known that $4xy \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5}z - \frac{1}{3\cdot5\cdot7}z^2 - \frac{1}{5\cdot7\cdot9}z^3$, &c. = area of the hyperbola.)

But $\frac{ty}{\sqrt{ta+x^2}} = c$ = conjugate axe, by the nature of the hyperbola. Conſequently the expreſſion for the rule =

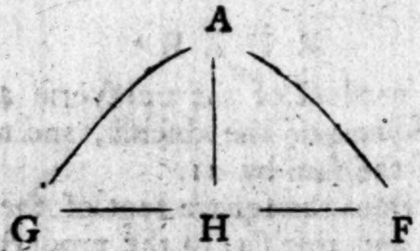
$$\frac{4cx}{t} \times \frac{21\sqrt{tx+\frac{5}{7}x}+4\sqrt{tx}}{75} = 4xy \times \frac{21\sqrt{tx+\frac{5}{7}x^2}+4\sqrt{tx}}{\sqrt{tx-x^2}}$$

And this thrown into a ſeries will very nearly agree with the former; which ſhews the rule to be an approximation.

Q. E. I.

Rule 2. If $2Y$, $2y$ = baſes, V, and v their diſtances from the centre, and the other letters as before, then will $VY - vy - \frac{tc}{4} \times$ hyp. log. of $\frac{Y+cv}{ty+cv}$ = area of the fruſtum of the hyperbola.

Rule



Here $21\sqrt{30\times10+\frac{5}{7}\times10^2}=21\sqrt{300+500\div7}=21\sqrt{300+71.42857}=21\sqrt{371.42857}=21\times19.272=404.712$.

And $(4\sqrt{30\times10}+404.712)\div75=(4\sqrt{400}+404.712)\div75=(4\times17.3205+404.712)\div75=(69.282+404.712)\div75=473.994\div75=6.3199$.

Whence $\frac{18\times19\times4}{30}\times6.3199=\frac{18\times4}{3}\times6.3199=6\times4\times6.3199=24\times6.3196=151.6776=$ *area required.*

2. The tranſverſe diameter is 100, the conjugate 60, and the leſſer abſciſs 50; what is the area of the hyperbola? *Anſ.* 3220.363472.

3. Required the area of the hyperbola to the abſciſſa 25, the two axes being 50 and 30. *Anſ.* 805.0909.

Rule 3. If t be put $=$ tranſverſe axis, $c=$ conjugate, and $x=$ abſciſſa, the area of a ſegment of an hyperbola, cut off by a double ordinate, will be $=\frac{4\sqrt{tx+\frac{5}{7}x^2}+\sqrt{tx}}{15}\times\frac{4cx}{t}$ extremely near.

OF

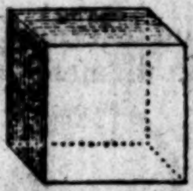
OF THE MENSURATION OF SOLIDS.

DEFINITIONS.

1. THE *meaſure* of any ſolid body, is the whole capacity or content of that body, when conſidered under the triple dimenſions of length, breadth, and thickneſs.

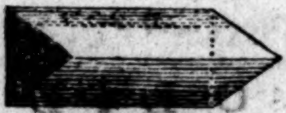
2. A *cube* whoſe ſide is one inch, one foot, or one yard, &c. is called the *meaſuring unit*; and the content or ſolidity of any figure, is computed by the number of thoſe cubes contained in that figure.

3. A *cube* is a ſolid contained by ſix equal ſquare ſides.



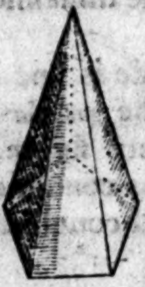
4. A *parallelopipedon* is a ſolid contained by ſix quadrilateral planes, every oppoſite two of which are equal and parallel.

5. A *priſm* is a ſolid whoſe ends are two equal, parallel, and ſimilar plane figures, and its ſides parallelograms.



6. A *cylinder* is a ſolid deſcribed by the revolution of a right angled parallelogram about one of its ſides, which remains fixed.

7. A * *pyramid* is a ſolid whoſe ſides are all triangles meeting in a point at the vertex, and the baſe any plane figure whatever.



8. A *ſphere* is a ſolid deſcribed by the revolution of a ſemi-circle about its diameter, which remains fixed.

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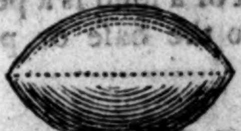
* The definition of a cone has been given already.

9. The centre of a ſphere is a point within the figure, every where equally diſtant from the convex ſurface of it.

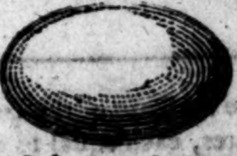
10. The diameter of the ſphere is a ſtraight line paſſing through the centre, and terminated both ways by the convex ſuperficies.

11. A *circular ſpindle* is a ſolid generated by the revolution of a ſegment of a circle about its chord, which remains fixed.



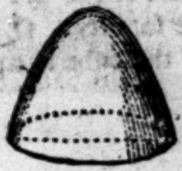
12. A *ſpheroid* is a ſolid generated by the revolution of a ſemi-ellipſis about one of its diameters, which is conſidered as quieſcent.

The ſpheroid is called *prolate*, when the revolution is made about the tranſverſe diameter, and *oblate* when it is made about the conjugate diameter.



13. *Elliptic, parabolic,* and *hyperbolic ſpindles,* are generated in the ſame manner as the circular ſpindle, the double ordinate of the ſection being always fixed or quieſcent.

14. *Parabolic* and *hyperbolic conoids,* are ſolids formed by the revolution of a ſemi-parabola or hyperbola about its tranſverſe axis, which is conſidered as quieſcent.



15. The *ſegment* of a pyramid, ſphere, or any other ſolid, is a part cut off from the top by a plane parallel to the baſe of that figure.

16. A *fruſtum* or *trunk*, is the part that remains at the bottom after the ſegment is cut off.

17. The *zone of a ſphere*, is that part which is intercepted between two parallel planes; and when thoſe planes are equally diſtant from the centre, it is called the middle zone of the ſphere.

18. The height of a ſolid is a perpendicular, drawn from its vertex to the baſe or plane on which it is ſuppoſed to ſtand.

PROBLEM I.

To find the ſolidity of a cube, the height of one of its ſides being given.

RULE.*

Multiply the ſide of the cube by itſelf, and that product again by the ſide, and it will give the ſolidity required.

---

* *Demon.* Conceive the baſe of the cube to be divided into a number of little ſquares, each equal to the *ſuperficial meaſuring unit*.

Then will thoſe ſquares be the baſes of a like number of ſmall cubes, which are each equal to the *ſolid meaſuring unit*.

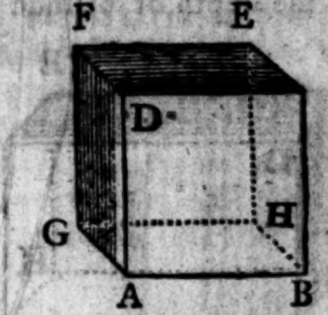
But the number of little ſquares contained in the baſe of the cube are equal to the ſquare of the ſide of that baſe, as has been ſhewn already.

And conſequently, the number of ſmall cubes contained in the whole figure, muſt be equal to the ſquare of the ſide of the baſe multiplied by the height of that figure; or, which is the ſame thing, the ſquare of the ſide of the baſe multiplied by the baſe, is equal to the ſolidity. Q. E. D.

Note. The ſurface of the cube is equal to 6 times the ſquare of its ſide.

EXAMPLES.

8. The ſide AB, or BC of the cube ABCDFGHE, is 25.5: what is the ſolidity?



Here $AB^3 = \overline{25.5|}^3 = 25.5 \times 25.5 \times 25.5 = 25.5 \times 650.25 = 16581.375$ *the anſwer, or content of the cube.*

2. The ſide of a cube is 15 inches: what is the ſolidity?

fo. in. pa.
Anſ. 1 11 5

3. What is the ſolidity of a cube whoſe ſide is 17.5 inches? *Anſ.* 3.101 *feet.*

PROBLEM II.

To find the ſolidity of a parallelopipedon.

RULE.*

Multiply the length by the breadth, and that product again by the depth or altitude, and it will give the ſolidity required.

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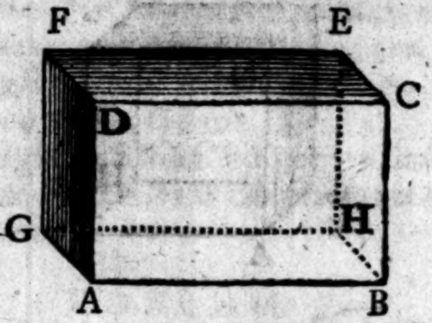
[*] The reaſon of this rule, as well as of the following ones for the priſm and cylinder, is the ſame as that for the cube.

Note. The ſurface of the parallelopipedon is equal to the ſum of the areas of each of its ſides or ends.

EXAMPLES.

1. Required the ſolidity of the parallelopipedon ABCDFEHG, whoſe length AB is 8 feet, its breadth FD $4\frac{1}{2}$ feet, and the depth or altitude AD $6\frac{3}{4}$ feet?



Here AB × AD × FD = 8 × 6.75 × 4.5 = 54 × 4.5 = 243 *ſolid feet, the contents of the parallelopipedon required.*

2. The length of a parallelopipedon is 15 feet, and each ſide of its ſquare baſe 21 inches: what is the ſolidity? *Anſ.* 45.9 *feet.*

3. What is the ſolidity of a block of marble, whoſe length is 10 feet, its breadth $5\frac{1}{4}$ feet, and the depth $3\frac{1}{2}$ feet? *Anſ.* 201.25 *feet.*

PROBLEM III.

To find the ſolidity of a priſm.

RULE.*

Multiply the area of the baſe into the perpendicular height of the priſm, and the product will be the ſolidity.

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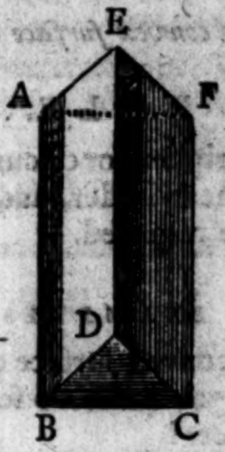
* The ſurface of a priſm, is equal to the ſum of the areas of the two ends and each of its ſides.

EXAMPLES.

1. What is the ſolidity of the triangular priſm ABCFED, whoſe length AB is 10 feet, and either of the equal ſides BC, CD, or DB, of one of its equilateral ends BCD, $2\frac{1}{2}$ feet?



*Here $\frac{1}{4} \times 2.5^2 \times \sqrt{3} = \frac{1}{4} \times 6.25 \times \sqrt{3} = 1.5625 \times \sqrt{3} = 1.5625 \times 1.732 = 2.70625 =$ *area of the baſe* BCD.

Or, $\frac{2.5 + 2.5 + 2.5}{2} = \frac{7.5}{2} = 3.75 = \frac{1}{2}$ *ſum of the ſides,* BC, CD, DB *of the triangle* CDB.

And $3.75 - 2.5 = 1.25$, ∴ 1.25. 1.25 *and* 1.25 = 3 *differences.*

Whence $\sqrt{3.75 \times 1.25 \times 1.25 \times 1.25} = \sqrt{3.75 \times 1.25^3} = \sqrt{7.32521875} = 2.7063 =$ *area of the baſe as before.*

And $2.7063 \times 10 = 27.063$ *ſolid feet, the content of the priſm required.*

2. What is the ſolidity of a triangular priſm, whoſe length is 18 feet, and one ſide of the equilateral end $1\frac{1}{2}$ feet? *Anſ.* 17.50859 *feet.*

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* This rule may be ſeen in the Notes to Problem III, p. 48.

2. Required the ſolidity of a priſm whoſe baſe is a hexagon, ſuppoſing each of the equal ſides to be 1 foot 4 inches, and the length of the priſm 15 feet.

Anſ. 69.282 *feet.*

PROBLEM IV.

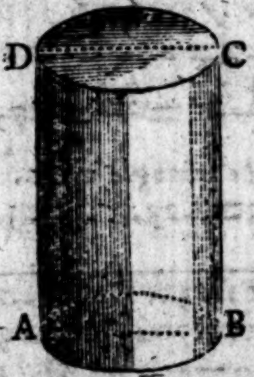
To find the convex ſurface of a cylinder.

RULE.*

Multiply the periphery or circumference of the baſe, by the height of the cylinder, and the product will be the convex ſurface required.

EXAMPLES.

1. What is the convex ſurface of the right cylinder ABCD, whoſe length BC is 20 feet, and the diameter of its baſe AB 2 feet?



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* *Demon.* If the periphery of the baſe be conceived to move, in a direction parallel to itſelf, it will generate the convex ſuperficies of the cylinder; and therefore the ſaid periphery being multiplied by the length of the cylinder, will be equal to that ſuperficies. Q. E. D.

Note. If twice the area of either of the ends be added to the convex ſurface, it will give the whole ſurface of the cylinder.

Here $3.14159 \times 2 = 6.28318 =$ *periphery of the baſe* A. B.

And $6.28318 \times 20 = 125.6636$ *ſquare feet, the convexity required.*

2. What is the convex ſurface of a right cylinder, the diameter of whoſe baſe is 30 inches, and the length 60 inches? *Anſ.* 5654.862 *inches.*

3. Required the convex ſuperficies of a right cylinder, whoſe circumference is 8 feet 4 inches, and its length 14 feet. *Anſ.* 116.666 *&c. feet.*

PROBLEM V.

To find the ſolidity of a cylinder.

RULE.[*]

Multiply the area of the baſe by the perpendicular height of the cylinder, and the product will be the ſolidity.

EXAM-

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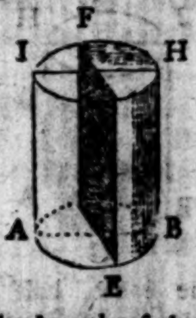
[*] The four following caſes contain all the rules for finding the ſuperficies, and ſolidities of *cylindric ungulas.*

I. *When the ſection is parallel to the axis of the cylinder.*



Rule. 1. Multiply the length of the arc line of the baſe by the height of the cylinder, and the product will be the *curve ſurface.*

2. Multiply the area of the baſe by the height of the cylinder, and the product will be the *ſolidity.*

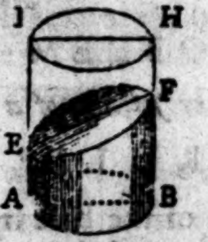
II. *When*

EXAMPLES.

1. What is the ſolidity of the cylinder ABCD, the diameter of whoſe baſe AB is 30 inches, and the height BC 50 inches?

Here

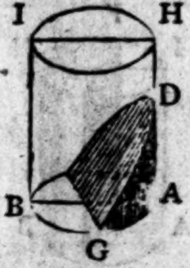
II. *When the ſection paſſes obliquely through the oppoſite ſides of the cylinder.*



Rule. 1. Multiply the circumference of the baſe of the cylinder by half the ſum of the greateſt and leaſt length, of the ungula, and the product will be the *curve ſurface.*

2. Multiply the area of the baſe of the cylinder by half the ſum of the greateſt and leaſt lengths of the ungula, and the product will be the *ſolidity.*

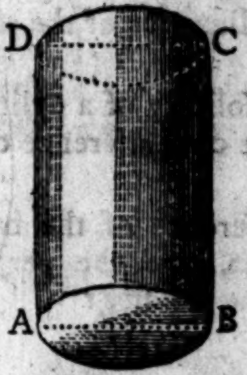
III. *When the ſection paſſes through the baſe of the cylinder, and one of its ſides.*



Rule. 1. Multiply the ſine of half the arc of the baſe by the diameter of the cylinder, and from this product ſubtract the product of the arc and coſine.

2. Multiply the difference, thus found, by the quotient of the height divided by the verſed ſine, and the product will be the *curve ſurface.*

1. From



Here $.7854\times30^2=.7854\times900=706.86=$ *area of the baſe* AB.

And $706.86\times50=35343$ *cubic inches; or* $\frac{35343}{1728}$ $=20.4531$ *ſolid feet, the anſwer required.*

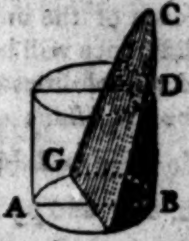
2. What

---

1. From $\frac{2}{3}$ of the cube of the right ſine of half the arc of the baſe, ſubtract the product of the area of the baſe and the coſine of the ſaid half arc.

2. Multiply the difference, thus found, by the quotient ariſing from the height divided by the verſed ſine, and the product will be the *ſolidity*.

IV. *When the ſection paſſes obliquely through both ends of the cylinder.*



Rule. 1. Conceive the ſection to be continued, till it meets the ſide of the cylinder produced; then ſay, as the difference of

2. What is the ſolidity of a cylinder whoſe height is 5 feet, and the diameter of the end 2 feet? *Anſ.* 15.708 *feet.*

3. What is the ſolidity of a cylinder whoſe height is 20 feet, and the circumference of its baſe 20 feet alſo? *Anſ.* 636.64 *feet.*

4. The circumference of the baſe of an oblique cylinder is 20 feet, and the perpendicular height 19.318: what is the ſolidity? *Anſ.* 614.926 *feet.*

PROBLEM VI.

To find the convex ſurface of a right cone.

RULE.*

Multiply the circumference of the baſe by the ſlant height, or the length of the ſide of the cone, and half the product will be the ſurface required.

EXAM-

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of the verſed ſines of half the arcs of the two ends of the ungula, is to the verſed ſine of half the arc of the leſſer end, ſo is the height of the cylinder to the part of the ſide produced.

2. Find the ſurface of each of the ungulas, thus formed, by caſe the third, and their difference will be the *ſurface required.*

3. In like manner find the ſolidities of each of the ungulas, and their difference will be the *ſolidity required.*

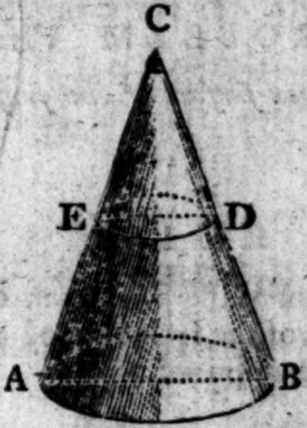
* *Demon.* Let AB $= a$, BC $= b$, $3.1416 = p$, and ED $= y$.

Then $a : b :: y : \frac{by}{a} =$ DC; and $py =$ circumference of the circle ED.

But

EXAMPLES.

1. The diameter of the baſe AB is 3 feet, and the ſlant height AC or BC 15 feet: required the convex ſurface of the cone ACB.



Here $3.1416\times3=9.4248=$ *circumference of the baſe* AB.

And, $\frac{9.4248\times15}{2}=\frac{141.3720}{2}=70.686$ *ſquare feet, the convex ſurface required.*

2. The diameter of a right cone is 4.5 feet, an the ſlant height 20 feet: required the convex ſurface.

Anſ. 141.372 *feet.*

But $py\times\frac{b\dot{y}}{a}=$ fluxion of the ſurface of CED, and its fluent $=\frac{pby^2}{2a}$; which, when $y=a$, becomes $\frac{pba}{2}=$ convex ſurface of the whole cone. Q. E. D.

To find the ſurface of a right pyramid.

Rule. Multiply the perimeter of the baſe by the length of the ſide, or ſlant height of the cone, and half the product will be the ſurface required.

3. The circumference of the baſe is 10.75, and the ſlant height 18.25: what is the convex ſurface?

Anſ. 98.09375.

PROBLEM VII.

To find the convex ſurface of the fruſtum of a right cone.

RULE.*

Multiply the ſum of the perimeters of the two ends, by the ſlant height of the fruſtum, and half the product will be the ſurface required.

---

* *Demon.* Let the perimeter of the circle AB $= \mathrm{P}$, that of ED $= p$, BD $= b$, and the reſt as in the laſt problem.

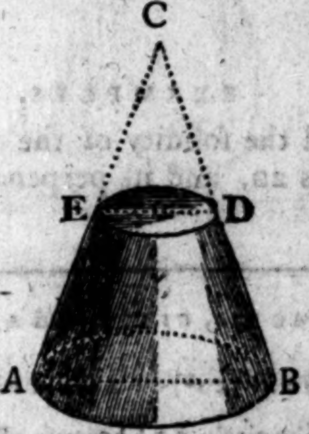
Then $\mathrm{P} : p :: b\,(\mathrm{BC}) : \mathrm{CD}$; and, by diviſion, $\mathrm{P} - p :: b - \mathrm{CD}\,(b) : \mathrm{CD} = \frac{pb}{\mathrm{P}-p}$; but $\mathrm{P} \times (b + \frac{pb}{\mathrm{P}-p}) =$ twice the convex ſurface of the whole cone, by the laſt rule; and alſo $p \times \frac{pb}{\mathrm{P}-p} =$ twice the convex ſurface of the part ECD. Therefore $\mathrm{P} \times (b + \frac{pb}{\mathrm{P}-p}) - p \times \frac{pb}{\mathrm{P}-p} = b\mathrm{P} + \overline{\mathrm{P}-p} \times \frac{pb}{\mathrm{P}-p} = b\mathrm{P} + bp = \overline{\mathrm{P}+p} \times b =$ twice the convex ſurface of the fruſtum ABDE; and the half thereof is $\frac{(\mathrm{P} \times p) \times b}{2}$, which is the ſame as the rule. Q. E. D.

To find the ſurface of the fruſtum of a right pyramid.

Rule. Multiply the ſum of the perimeters of the end by the ſlant height, and half the product will be the ſurface required.

EXAMPLES.

1. In the fruſtum ABED, the circumferences of the two ends AB and DE are 22.5 and 15.75 reſpectively, and the ſlant height BD is 26: what is the convex ſurface?



Here $\frac{(22.5+15.75)\times 26}{2} = \overline{22.5+15.75}\times 13 =$ $38.25\times 15 = 497.25 =$ *convex ſurface required.*

2. What is the convex ſurface of the fruſtum of a right cone, the circumference of the greater end being 30 feet, that of the leſſer end 10 feet, and the length of the ſlant ſide 20 feet? *Anſ.* 400 *feet.*

3. What is the convex ſurface of the fruſtum of a right cone, the diameters of the ends being 8 and 4 feet, and the length of the ſlant ſide 20 feet? *Anſ.* 376.992 *feet.*

4. If a ſegment of 6 feet ſlant height be cut off a cone whoſe ſlant height is 30 feet, and circumference of its baſe 10 feet: what is the ſurface of the fruſtum? *Anſ.* 144 *feet.*

PROBLEM VIII.

To find the ſolidity of a cone or pyramid.

RULE.*

Multiply the area of the baſe by $\frac{1}{3}$ of the perpendicular height of the cone, and the product will be the ſolidity.

EXAMPLES.

1. Required the ſolidity of the cone ACB, whoſe diameter AB is 20, and its perpendicular height CS 24.

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* *Demon.* Let SC $= a$, C$s = x$, and A $=$ area of the baſe of the cone ACB.

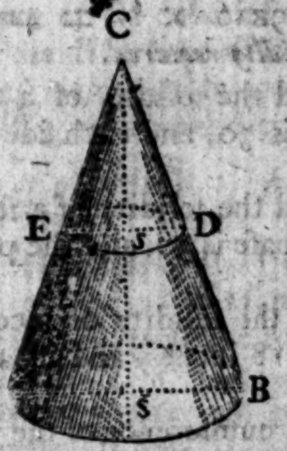
Then a^2 (CS^2) : x^2 (Cs^2) : : AB^2 : ED^2 (by ſim. $\triangle s$) : : A : $\frac{Ax^2}{a}$, ($=$ area of the circle ED) becauſe all the circles are as the ſquares of their diameters.

But $\frac{Ax^2}{a^2} \times \dot{x} =$ fluxion of the cone ECD, and its fluent $=$ $\frac{Ax^3}{3a^2}$; which, when $x = a$, becomes $\frac{Aa}{3} = A \times \frac{a}{3}$ for the ſolidity of the whole cone. Q. E. D.

In the pyramid CEDB it will be a^2 (CS^2) : x^2 (Cs^2) : : CE^2 : Ce^2 : : ED^2 : eo^2 (by ſim. $\triangle s$) : : A (area of the baſe EB) : $\frac{Ax^2}{a^2}$ (area of the polygon eb) becauſe all ſimilar figures are as the ſquares of their like ſides.

But $\frac{Ax^2}{a^2} \times \dot{x} =$ fluxion of the pyramid Ceob, and its correct fluent $= A \times \frac{a}{3}$ the ſame as in the cone; and this rule is general, let the figure of the baſe be what it will.

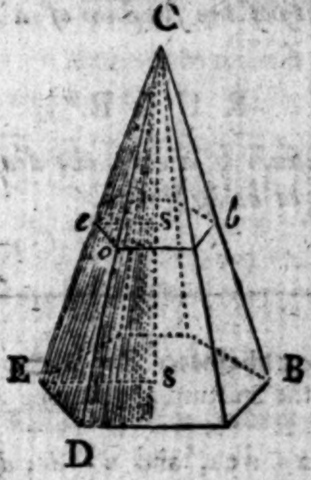
Here



Here $.7854 \times 20^2 = .7854 \times 400 = 314.16 =$ *area of the baſe* A B.

And $314.16 \times \frac{24}{3} = 314.16 \times 8 = 2513.28 =$ *ſolidity required.*

2. Required the ſolidity of the hexagonal pyramid E C B D, each of the equal ſides of its baſe being 40, and the perpendicular height C S 60.



Here 2.598076 (*multiplier when the ſide is* 1) $\times 40^2 =$ $2.598076 \times 1600 = 4156.9216 =$ *area of the baſe.*

And, $4156.9216 \times \frac{60}{3} = 4156.9216 \times 20 = 83138.432$ *ſolidity required.*

3. Required the ſolidity of a triangular pyramid, whoſe height is 30, and each ſide of the baſe 3.

Anſ. 38.97117.

4. Required the ſolidity of a ſquare pyramid, each ſide of whoſe baſe is 30, and the perpendicular height 20. *Anſ.* 6000.

5. What is the ſolidity of a cone, the diameter of whoſe baſe is 18 inches, and its altitude 15 feet?

Anſ. 8.83575 *feet.*

6. If the circumference of the baſe of a cone be 40 feet, and the height 50 feet; what is the ſolidity?

Anſ. 2120 *feet.*

7. What is the content of a pentagonal pyramid, its height being 12 feet, and each ſide of its baſe 2 feet?

Anſ. 27.527.

PROBLEM IX.

To find the ſolidity of the fruſtum of a cone or pyramid.

RULE.*

1. *For the fruſtum of a cone, the diameters of the two ends, and the height being given.*

Divide

---

* *Demon.* Let $D =$ diameter AB, $d =$ ED, $p = .7854$, and $h =$ S*s* $=$ height of the fruſtum of the cone ABDE, (ſee the laſt figure.)

Then $D : d :: BS : Cs$, and $D - d : d :: CS - Cs\ (h) : \frac{dh}{D-d} = Cs =$ height of the cone EDC.

But

Divide the difference of the cubes of the diameters of the two ends, by the difference of the diameters, and this quotient being multiplied by .7854 and again by $\frac{1}{3}$ of the height will give the ſolidity.

2. *For the fruſtum of a pyramid, the ſides and height being given.*

To the areas of the two ends of the fruſtum add the ſquare root of their product, and this ſum being multiplied by $\frac{1}{3}$ of the height will give the ſolidity.

1. What is the ſolidity of the fruſtum of the cone EABD, the diameter of whoſe greater end AB is

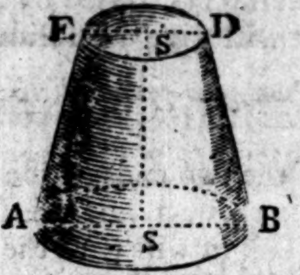
---

But $\frac{pD^2}{3} \times (b + \frac{db}{D-d}) =$ ſolidity of the whole cone ACB, and $\frac{pd^2}{3} \times \frac{db}{D-d} =$ that of the cone ECD. Therefore $\frac{pD^2}{3} \times (b + \frac{db}{D-b}) - \frac{pd^2}{3} \times \frac{db}{D-d} = (D^2 \times \frac{Db}{D-d} - d^2 \times \frac{db}{D-d}) \times \frac{p}{3} = (bD^2 + \overline{D^2-d^2} \times \frac{db}{D-d}) \times \frac{p}{3} = \frac{D^3-d^3}{D-d} \times \frac{bp}{3} =$ ſolidity of the fruſtum ABDE. Q. E. D.

Again, for the polygon, let $S = ED$, $s = ed$, and $m =$ proper multiplier in the table of polygons; then $S : s :: CS : Cs$, and $S - s : s :: CS - Cs\ (b) : \frac{bs}{S-s}$.

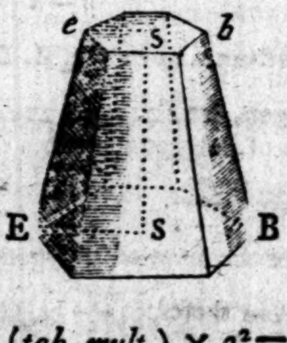
But mS^2 and ms^2 are the areas of polygons whoſe ſides are S and s reſpectively. And therefore $\frac{mS^2}{3} \times (b + \frac{bs}{S-s}) - \frac{ms^2}{3} \times \frac{bs}{S-s} = (mS^2 + \overline{mS^2 - ms^2} \times \frac{bS}{S-s} \times \frac{b}{3} = (ms + msS + ms^2) \times \frac{b}{3} =$ ſolidity of the fruſtum $eEDBb$, which is the ſame as the rule.

5 feet, that of the leſſer end ED 3 feet, and the perpendicular height *s*S 9 feet?



Here $\frac{5^3-3^3}{5-3}\times .7854\times\frac{9}{3}=\frac{125-27}{2}\times .7854\times 3=$ $\frac{98}{2}\times 2.3562=49\times 2.3562=115.4538$ *ſolid feet, the content of the fruſtum.*

2. What is the ſolidity of the fruſtum *e*EDB*b*, of an hexagonal pyramid, the ſide ED of whoſe greater end is 4 feet, that *ed* of the leſſer end 3 feet, and the height S*s* 9 feet?



Here 2.598076 (*tab. mult.*) $\times 3^2=2.598076\times 9=23.382624=$ *area of the polygon eb.*

And 2.598076 (*tab. mult.*) $\times 4^2=2.598076\times 16=41.569216=$ *area of the polygon* EB.

Whence $\sqrt{41.569216 \times 23.382684} = \sqrt{971.999841}$ $= 31.176911$.

And $(41.569216 + 23.382684 + 31.176911) \times \frac{9}{3} =$ $96.128811 \times 3 = 288.386433$ *feet*, = *ſolidity required.*

3. What is the ſolidity of the fruſtum of a cone, the diameter of the greater end being 4 feet, that of the leſſer end 2 feet, and the altitude 9 feet?

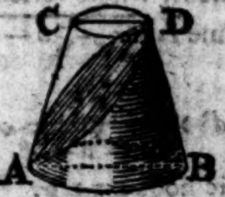
Anſ. 65.9736.

4. What

---

[*] The following caſes contain all the rules for finding the ſuperficies and ſolidities of conical ungulas.

1. *When the ſection paſſes through the oppoſite extremities of the ends of the fruſtum.*

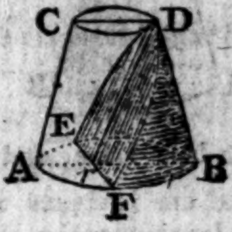


Let D = A B, the diameter of the greater end, d = C D the diameter of the leſs end, b = perpendicular height of the fruſtum, and $n = .78539$, &c.

Then will $\overline{D^2 - d\sqrt{Dd}} \div \overline{D - d} \times \frac{1}{3} n d b$ = ſolidity of the elliptic ungula A D B.

And $\frac{n}{D - d} \sqrt{\overline{4b^2 + (D - d^2)}} \times \overline{D^2 - (D - d)\sqrt{4Dd}}$ = curve ſurface of A D B.

2. *When the ſection cuts off part of the baſe, and makes the angle D r B leſs than the angle C A B.*



Let

4. What is the ſolidity of the fruſtum of a cone, the circumference of the greater end being 40, that of the leſſer end 20, and the length or height 50?

Anſ. 3713.64.

5. What is the ſolidity of the fruſtum of a ſquare pyramid, one ſide of the greater end being 18 inches, that of the leſſer end 15 inches, and the height 60 inches? *Anſ.* 16380 *inches.*

6. What is the ſolidity of the fruſtum of an hexagonal pyramid, the ſide of whoſe greater end is 3 feet, that of the leſſer end 2 feet, and the length 12 feet?

Anſ. 197.453776 *feet.*

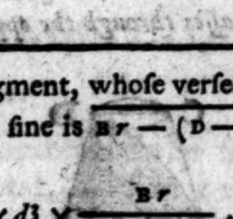
PRO-

Let s = tabular ſegment, whoſe verſed ſine is $Br \div D$, s = tab. ſeg. whoſe verſed ſine is $\overline{Br - (D - d)} \div d$, and the other letters as before.

Then $\left(s \times D^3 - s \times d^3 \times \frac{Br}{Br - \overline{D - d}} \sqrt{\frac{Br}{Br - \overline{D - d}}}\right) \times \frac{\frac{1}{3}b}{D - d}$ = ſolidity of the elliptic hoof EFBD.

And $\frac{1}{D - d}\sqrt{4b^2 + (D - d)^2} \times$ (ſeg. FBE $- \frac{d^2}{D^2} \times \frac{\frac{1}{2} \times (D + d) - Ar}{d - Ar} \times \sqrt{\frac{Br}{d - Ar}} \times$ ſeg. of the circle AB, whoſe height is $D \times \frac{d - Ar}{d}$ = convex ſurface of EFBD.

3. *When the ſection is parallel to one of the ſides of the fruſtum.*



Let A = area of the baſe FBE, and the other letters as before.

Then

PROBLEM X.

To find the ſolidity of a cuneus or wedge.

RULE.*

Multiply the ſum of twice the length of the baſe and the length of the edge, by the product of the height

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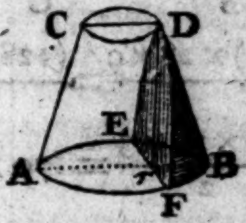
Then $\left(\frac{A\times D}{D-d}-\frac{4}{3}d\sqrt{(D-d)\times d}\right)\times\frac{1}{2}b=$ ſolidity of the parabolic hoof EFBD.

And $\frac{1}{D-d}\sqrt{4b^2+(D-d)^2}\times(\text{ſeg. FBE}-\frac{2}{3}\overline{D-d}\times\sqrt{d\times\overline{D-d}})=$ convex ſurface of EFBD.

4. *When the ſection cuts off part of the baſe, and makes the angle DrB greater than the angle CAB.*



Let the area of the hyperbolic ſection EDF $=$ A, and the area of the circular ſeg. EBF $=a$.

Then $\frac{\frac{1}{8}b}{D-b}\times\left(a\times D-A\times\frac{d\times Br}{Cr}\right)=$ ſolidity of the hyperbolic ungula EFBD.

And $\frac{1}{D-d}\times\sqrt{4b^2+(D-d)^2}\times(\text{cir. ſeg. EBF}-\frac{d^2}{D^2}\times\frac{Br-\frac{1}{2}(D-d)}{Br-\overline{D-d}}\sqrt{\frac{Br}{Br-\overline{D-d}}}=$ curve ſurface of EFBD.

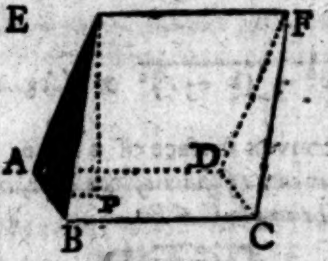
Note. The tranſverſe diameter of the hyp. ſeg. $=\frac{d\times Cr}{D-d-Br}$ and the conjugate $=d\sqrt{\frac{Br}{D-d-Br}}$, from which its area may be found by the former rules.

* *Demon.* When the length of the baſe is equal to that of the wedge, the wedge is evidently equal to half a priſm of the ſame baſe and altitude. And

height of the wedge and the breadth of the baſe, and $\frac{1}{6}$ of the laſt product will be the ſolidity.

EXAMPLES.

1. How many ſolid feet are there in a wedge, whoſe baſe is 5 feet 4 inches long, and 9 inches broad, the length of the edge being 3 feet 6 inches, and the perpendicular height 2 feet 4 inches?



Here $\frac{(64\times2+42)\times28\times9}{6}=\frac{(128+42)\times28\times9}{6}=$ $\frac{170\times28\times9}{6}=\frac{170\times28\times3}{2}=170\times14\times3=7140$ *ſolid inches.*

And according as the edge is ſhorter or longer than the baſe, the wedge is greater or leſs than half a priſm, by a pyramid of the ſame height and breadth at the baſe with the wedge, and the length of whoſe baſe is equal to the difference of the lengths of the edge and baſe of the wedge.

Therefore, let the length of the baſe BC $=L$; the length of the edge EF $=l$; the breadth of the baſe BA $=b$; and the height of the wedge EP $=b$; and we ſhall have by the former rules $\frac{blb}{2}\pm bb\times\frac{\pm L\mp l}{3}=\frac{blb}{2}+bb\times\frac{L-l}{3}=bb\times$ $\frac{3l+2L-2l}{2}=bb\times\frac{2L+l}{6}$. Q. E. D.

And 7140 ÷ 1728 = 4.1319 *ſolid feet, the content required.*

2. The length and breadth of the baſe of a wedge are 35 and 15 inches, and the length of the edge is 55 inches: what is the ſolidity, ſuppoſing the perpendicular height to be 17.14508 inches?

Anſ. 3.1006 *feet.*

PROBLEM XI.

To find the ſolidity of a priſmoid.

RULE.*

To the ſum of the areas of the two ends add four times the area of a ſection parallel to and equally diſtant from both ends, and this laſt ſum multiplied by $\frac{1}{6}$ of the height will give the ſolidity.

Note. The length of the middle rectangle is equal to half the ſum of the lengths of the rectangles of the two ends, and its breadth equal to half the ſum of the breadths of thoſe rectangles.

EXAM-

---

* *Demon.* The rectangular priſmoid is evidently compoſed of two wedges, whoſe heights are equal to the height of the priſmoid, and their baſes its two ends. Wherefore, by the laſt problem, its ſolidity will be $= (\overline{2L + l} \times B + \overline{2l + L} \times b) \times \frac{h}{6}$, which, by putting $M = \frac{L + l}{2}$, and $m = \frac{B + b}{2}$, becomes $\overline{BL + bl + 4Mm} \times \frac{h}{6}$; which is the rule, as was to be ſhewn.

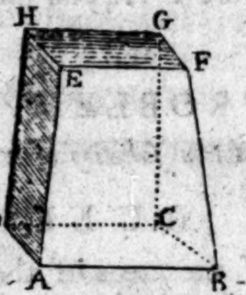
The ſolidities of the two parts, commonly called the ungulas, or hoofs, into which the fruſtum of a rectangular pyramid is divided, may be found by the two laſt rules, as they are only compoſed of wedges and priſmoids.

A very elegant demonſtration of this rule for the priſmoid may be ſeen in *Simpſon*'s *Fluxions, page* 179, 2*d Edition.*

If

EXAMPLES.

1. What is the ſolidity of a rectangular priſmoid the length and breadth of one end being 14 and 12 inches, and the correſponding ſides of the other 6 and 4 inches; and the perpendicular $30\frac{1}{2}$ feet?



Here $14\times12+6\times4=168+24=192=$ *ſum of the areas of the two ends.*

Alſo $\frac{14+6}{2}=\frac{20}{2}=10=$ *length of the middle rectangle.*

And $\frac{12+4}{2}=\frac{16}{2}=8=$ *breadth of the middle rectangle.*

Whence $10\times8\times4=80\times4=320=4$ *times the area of the middle rectangle.*

Or $(320+192)\times\frac{366}{6}=512\times61=31232$ *ſolid inches.*

And $31242\div1728=18.074$ *ſolid feet, the content.*

2. What is the ſolid content of a priſmoid, whoſe greater end meaſures 12 inches by 8, the leſſer end 8 inches by 6, and the length, or height, 60 inches?

Anſ. 2.453 *feet.*

If the baſes of the priſmoid are diſſimilar rectangles, of which L, l and M, m are correſponding dimenſions, and h the height; Then $(\overline{2L+l}.M+\overline{2l+L}.m)\times\frac{1}{6}h=$ ſolidity.

3. What is the capacity of a coal waggon, whoſe inſide dimenſions are as follows: at the top, the length is 81½, and breadth 55 inches; at the bottom, the length is 41, and the breadth 29½ inches; and the perpendicular depth is 47¼ inches?

Anſ. 126340.59375 *cubic inches; which is nearly equal to a chaldron of coals.*

PROBLEM XII.

To find the convex ſurface of a ſphere.

RULE.*

Multiply the diameter of the ſphere by its circumference, and the product will be the convex ſuperficies required.

Note. The curve ſurface of any zone or ſegment will alſo be found by multiplying its height by the whole circumference of the ſphere.

EXAM-

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* *Demon.* Put the diameter BC $= d$, BA $= x$, AC $= y$, BC $= z$; and 3.14159, &c. $= p$.

Then, ſince the triangles AOC and CED are ſimilar, we ſhall have CA (y) : CO $(\frac{d}{2})$: : CE $(\dot{x})$: CD $(\dot{x}) = \frac{d\dot{x}}{2y}$. But $2py\dot{z}$ is the general expreſſion for the fluxion of any ſurface; and therefore by ſubſtituting $\frac{d\dot{x}}{2y}$ for its equal $\dot{z}$, the fluxion will become $pd\dot{x}$; and conſequently $pdx =$ ſurface of any ſegment of a ſphere whoſe height is x, and $pdd =$ that of the whole ſphere. Q. E. D.

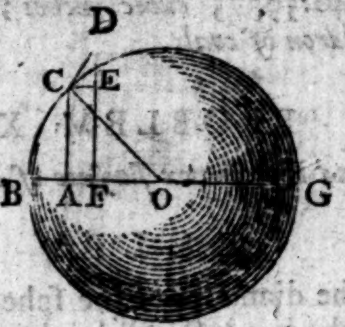
Cor. 1. The ſurface of the ſphere is equal to the curve ſurface of its circumſcribing cylinder.

Cor. 2. The ſurface of a ſphere is alſo equal to 4 times the area of a great circle of it.

1. To

EXAMPLES.

1. What is the convex ſuperficies of a globe BCG whoſe diameter BG is 17 inches?



Here 3.14159 × 17 × 17 = 53.40703 × 17 = 907.91951 *ſquare inches.*

And, 907.91951 ÷ 144 = 6.30499 *ſquare feet the anſwer.*

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1. To find the lunar ſurface included between two great circles of the ſphere.

Rule. Multiply the diameter into the breadth of the ſurface in the middle, and the product will be the ſuperficies required.

Or, As one right angle is to a great circle of the ſphere;
So is the angle made by the two great circles,
To the ſurface included by them.

2. To find the area of a ſpherical triangle, or the ſurface included by the interſecting arcs of three great circles of the ſphere.

Rule. As two right angles, or 180°,
Is to a great circle of the ſphere;
So is the exceſs of the three angles above two right angles,
To the area of the triangle.

2. What is the convex ſuperficies of a ſphere whoſe diameter is $1\frac{1}{3}$ feet, and the circumference 4.1888 feet? *Anſ.* 5.58506 *feet.*

3. If the diameter, or axis of the earth, be $7957\frac{3}{4}$ miles; what is the whole ſurface, ſuppoſing it to be a perfect ſphere? *Anſ.* 198943653 *ſquare miles.*

4. The diameter of a ſphere is 21 inches; what is the convex ſuperficies of that ſegment of it whoſe height is $4\frac{1}{2}$ inches? *Anſ.* 296.8802 *inches.*

5. What is the convex ſurface of a ſpherical zone, whoſe breadth is 4 inches, and the diameter of the ſphere, from which it was cut, 25 inches?
Anſ. 314.16 *inches.*

PROBLEM XIII.

To find the ſolidity of a ſphere or globe.

RULE.*

Multiply the cube of the diameter by .5236, and the product will be the ſolidity.

EXAM-

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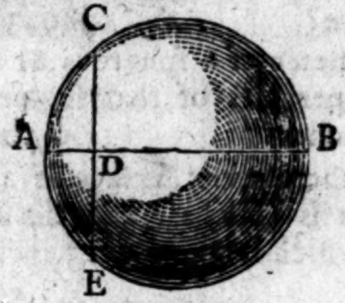
* *Demon.* Put AD $= x$, CD $= y$, the diameter AB $= d$, and $p = 3.14159$.

Then, by the property of the circle, $dx - x^2 = y^2$. But the general expreſſion for the fluxion of any ſolid is $p y^2 \dot{x}$; and therefore by writing $dx - x^2$ for its equal y^2, we ſhall have $p\dot{x} \times \overline{dx - x^2} = pdx\,\dot{x} - px^2\dot{x}$. The fluent of which is $\frac{pdx^2}{2} - \frac{px^3}{3} = \frac{3pdx^2 - 2px^3}{6} =$ content of the ſegment CAE.

H And

EXAMPLES.

1. What is the ſolidity of the ſphere A E B C, whoſe diameter A B is 17 inches?



Here $17^3 \times .5236 = 17 \times 17 \times 17 \times .5236 = 289 \times 17 \times .5236 = 4913 \times .5236 = 5272.4468$ *ſolid inches;*

And $5272.4468 \div 1728 = 1.48868$ *ſolid feet the anſwer.*

2. What is the ſolidity of a ſphere whoſe diameter is $1\frac{1}{3}$ feet? *Anſ.* 1.2411 *feet.*

3. What is the ſolidity of the earth, ſuppoſing it to be perfectly ſpherical, and its diameter $7957\frac{4}{3}$ miles? *Anſ.* 263858149120 *miles.*

PROBLEM XIV.

To find the ſolidity of the ſegment of a ſphere.

And if d be ſubſtituted for x it will become $\frac{3pd^3 - 2pd^3}{9} = \frac{pd^3}{6} = d^3 \times .5236$, or $.5236d^3$; which is the ſame as the rule.

Coroll. A ſphere, or globe, is equal to two-thirds of its circumſcribing cylinder.

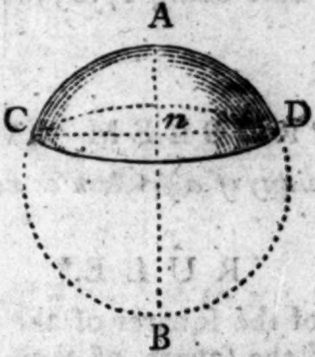
RULE.

RULE.*

To three times the ſquare of the radius of its baſe add the ſquare of its height; and this ſum multiplied by the height, and the product again by .5236 will give the ſolidity.

EXAMPLES.

1. The radius C *n* of the baſe of the ſegment C A D is 7 inches, and the height A *n* 4 inches: what is its ſolidity?



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* *Demon.* Let $r =$ radius of the baſe of the ſegment, $b =$ height of the ſegment, and the other letters as before.

Then will $(3db^2 - 2b^3) \times \frac{p}{6} =$ ſolidity of the ſegment, as is ſhewn in the laſt problem.

But ſince $\frac{r^2 + b^2}{b} = d$, by the property of the circle, we ſhall have $\left(\frac{3r^2b^2 + 3b^4}{b} - 2b^3\right) \times \frac{p}{6} = \overline{3r^2 + b^2} \times \frac{pb}{6} =$ ſolidity of the ſegment, which is the ſame as the rule.

Or if $d =$ diameter of the ſphere, and $b =$ height of the ſegment; then will $.5236b^2 \times (3d - 2b) =$ ſolidity.

Here $(7^2 \times 3 + 4^2) \times 4 \times .5236 = (49 \times 3 + 4^2) \times 4 \times .5236 = (147 + 4^2) \times 4 \times .5236 = (147 + 16) \times 4 \times .5236 = 163 \times 4 \times .5236 = 652 \times .5236 = 341.3872$ *ſolid inches, the anſwer.*

2. What is the ſolidity of the ſegment of a ſphere, the diameter of whoſe baſe is 20, and its height 9?

Anſ. 1799.6132.

3. What is the content of the ſpherical ſegment, whoſe height is 4 inches, and the radius of its baſe 8?

Anſ. 435.6352.

4. What is the ſolidity of a ſpherical ſegment, the diameter of its baſe being 17.23368, and its height 4.5?

Anſ. 572.5566.

PROBLEM XVI.

To find the ſolidity of a fruſtum or zone of a ſphere.

RULE.*

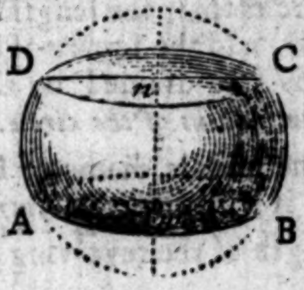
To the ſum of the ſquares of the radii of the two ends, add $\frac{1}{3}$ of the ſquare of their diſtance, or the breadth of the zone, and this ſum multiplied by the ſaid breadth, and the product again by 1.5708 will give the ſolidity.

---

* *Demon.* The difference between two ſegments of a ſphere whoſe heights are H and b, and the radii of whoſe baſes are R and r, will, by the laſt problem $= \frac{p}{6} \times (3R^2H + H^3 - 3r^2b - b^3)$ = zone whoſe height is $H - b$. And therefore by putting a for the altitude of the fruſtum, and exterminating H and b by the means of the two equations $\frac{3R^2 + H^2}{H} = \frac{r^2 + b^2}{b}$ and $H - b = a$, we ſhall have $(R^2 + r^2 + \frac{a^2}{3}) \times \frac{pa}{2}$, which is the rule,

EXAMPLES.

1. What is the ſolid content of the zone ABCD, whoſe greater diameter AB is 20 inches, the leſſer diameter CD 15 inches, and the diſtance *nm* of the two ends 10 inches?



Here $(10^2+7.5^2+\frac{10^2}{3})\times 10\times 1.5708=(100+56.25+33.33)\times 10\times 1.5708=189.58\times 10\times 1.5708$ $=1895.8\times 1.5708=2977.92264$ *ſolid inches, the anſwer.*

2. What is the ſolid content of a zone, whoſe greater diameter is 24 inches, the leſſer diameter 20 inches, and the diſtance of the ends 4 inches?

Anſ. 1566.6112 *inches.*

3. Required the ſolidity of the middle zone of a ſphere, whoſe top and bottom diameters are each 3 feet, and the breadth of the zone 4 feet.

Anſ. 61.7848 *feet.*

If it be the middle zone of the ſphere, the ſolidity will be $=(d^2+\frac{2}{3}b^2)\times .7854b$; where $d =$ diameter of each end, and $b =$ its height.

PROBLEM XVII.

To find the ſurface of a circular ſpindle, the length and breadth, or middle diameter, being given.

RULE.*

1. To the ſquare of half the length of the ſpindle, or longeſt diameter, add the ſquare of half the middle diameter, and this ſum divided by the middle diameter will give the *radius of the circle*.

2. Take half the middle diameter from the radius of the circle, and it will give the *central diſtance*.

3. Find the length of the revolving arc by problem the 10th.

4. From the product of the longeſt diameter and the radius of the revolving arc, ſubtract the product of the ſaid arc and the central diſtance, and this remainder multiplied by 6.2832 will give the ſurface required.

EXAM-

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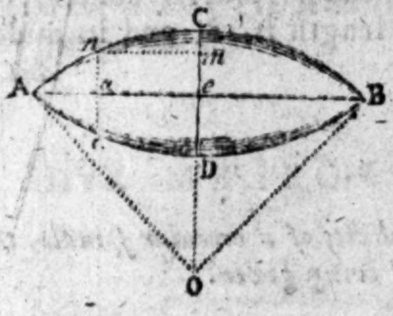
[*] *Demon.* Put $z =$ arc cn, $x =$ its ſine nm, $r =$ radius co, $c =$ central diſtance oe, and $p = 3.14159$, &c.

Then $\sqrt{r^2 - x^2}$ $(om) : r :: \dot{x} : \dot{z}$, and $\dot{z}\sqrt{r^2 - x^2} = r\dot{x}$, by the property of the circle, as is ſhewn in the demonſtration to problem 13. But $2p\dot{z} \times na = 2p\dot{z}(\sqrt{r^2 - x^2} - c) = 2p \times (\dot{z}\sqrt{r^2 - x^2} - c\dot{z})$ is the general expreſſion for the fluxion, and therefore $2p \times (r\dot{x} - c\dot{z}) =$ fluxion of half the fruſtum c$ne$D, and $2p \times (rx - cz) = 2p \times (r \times ae - c \times$ c$n) =$ to its ſurface. Q. E. D.

Coroll.

EXAMPLES.

1. What is the ſuperficial content of the circular ſpindle ADBC, whoſe length AB is 48, and the middle diameter CD 36?



Here $(24^2+18^2)\div 36=(576+324)\div 36=\frac{900}{36}=\frac{100}{4}=25=$ OC *the radius of the circle* ACB.

Therefore $25-18=7=$ OC $-$ *ce* $=$ *central diſtance* O*e*.

And $\frac{2\times 18}{3}\div\left(50-\frac{41\times 18}{50}\right)=12\div\overline{50-14.76}=12\div 35.24=.34052$.

Coroll. 1. When $ea=e\mathrm{A}$, the rule becomes $2p\times(r\times e\mathrm{A}-c\times \mathrm{CA})$ for the ſurface of half the ſpindle, and $2p\times\overline{r\times \mathrm{AB}-c\times \mathrm{ACB}}$ for that of the whole ſpindle.

Coroll. 2. If from the ſurface of the ſemi-ſpindle there be taken that of the fruſtum, there will remain $2p\times\overline{r\times a\mathrm{A}-c\times \mathrm{A}n}$ for the ſegment $n\mathrm{A}c$; ſo that the rule is general.

Coroll. 3. When *e* coincides with o, *c* vaniſhes, and the ſpindle becomes a ſphere; in which caſe the theorem alſo becomes barely $2prx$, the ſame expreſſion as was before found for the ſurface of the ſphere.

Whence $(1+.34052)\times 48=1.34052\times 48=64.34496$ $=$ *length of the arc* ACB.

And $(48\times 25-64.34496\times 7)\times 6.2832=(1200-450.41472)\times 6.2832=749.58528\times 6.2832=4709.794231296=$ *ſuperficies required.*

2. What is the ſuperficial content of a circular ſpindle whoſe length is 48, and its middle diameter 30? *Anſ.* 4387.1644.

PROBLEM XVIII.

To find the ſolidity of a circular ſpindle, the length and middle diameter being given.

RULE.*

1. Find the area of the generating circular ſegment by problem the 13th; and the radius and central diſtance as in the laſt problem.

2. From $\frac{1}{3}$ of the cube of half the length of the ſpindle ſubtract the product of the central diſtance

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* *Demon.* Put $ae=x$, $oe=c$, $r=$ radius, and $p=3.14159$, &c.

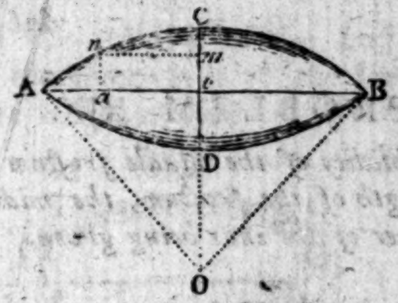
Then will the fluxion of the ſolid be $=p\dot{x}\times na^2=p\dot{x}\times(mo-c)^2=p\dot{x}\times mo^2-c\times\overline{2mo-c}=p\dot{x}\times mo^2-c\times\overline{2an+c}=p\dot{x}\times(r^2-x^2-c^2-2c\times an)$; whoſe fluent is $=px\times(r^2-\frac{x^2}{3}-c^2-\frac{2c}{x}\times ance)=p\times\overline{(Ae^2-\frac{1}{2}x^2\times x-2c\times ance)}=$ fruſtum generated by $ance$. And when $x=Ae$, we have $\overline{\frac{1}{3}Ae^3-c\times ACe}\times 2p$ for the half ſpindle CAD, and $\overline{\frac{1}{3}Ae^3-c\times ACe}\times 4p=$ whole ſpindle ADBC. Q. E. D.

Coroll.

and half the area of the generating ſegment, and this remainder multiplied by 12.5664 will give the ſolidity.

EXAMPLES.

1. The longeſt diameter AB of the circular ſpindle ADBC is 48, and the middle diameter CD 36: what is the ſolidity of the ſpindle?



Here $\frac{48 \times 7}{2} = 24 \times 7$ (*7 being the central diſtance* O*e*, *as in the laſt prob.*) $= 168 =$ *area of the triangle* AOB.

And 32.17248 ($\frac{1}{2}$ *length of the arc* ACB *by laſt prob.*) $\times$ 25 (*radius* OC *by ſame prob.*) $= 804.312 =$ *area of the ſector* BCAO.

Coroll. 1. If the fruſtum be taken from the half ſpindle there will remain $p \times (Ae^2 \times Aa - \frac{1}{3} \times \overline{Ae^3 - x^3} - 2c \times Ana) = p \times (\frac{1}{3} Aa^3 \times \overline{3Ae - Aa} - 2c \times Ana)$ for the ſegment of the ſpindle twice nAa.

Coroll. 2. When e coincides with o, c will vaniſh, and the theorem will become $\frac{4}{3} p \times oc^3 = \frac{1}{6} p \times D^3$, which is the ſolidity of the whole ſphere.

Which theorems agree with thoſe before given.

Alſo $804.312 - 168 = 636.312 =$ *area of the ſegment ACBE.*

And $(\frac{24^3}{3} - \frac{636.312}{2} \times 7) \times 12.5664 = (\frac{13824}{3} - 318.156 \times 7) \times 12.5664 = (4608 - 2227.092) \times 12.5664 = 2380.908 \times 12.5664 = 29919.4422912 =$ *ſolidity required.*

2. If the length of a circular ſpindle be 40, and its middle diameter 30: what is its ſolidity?

Anſ. 17310.458.

PROBLEM XIX.

To find the ſolidity of the middle fruſtum of a circular ſpindle, the length of the fruſtum, the middle diameter, and that of either of the ends being given.

RULE.*

1. Divide the ſquare of half the length of the fruſtum by half the difference of the middle diameter, and that of either of the two ends; and half this quotient added to $\frac{1}{4}$ of the ſaid difference will give the radius of the circle.

2. Find the central diſtance, and the revolving area, as in the laſt problem.

3. From the ſquare of the radius take the ſquare of the central diſtance, and the ſquare root of the remainder will give half the length of the ſpindle.

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* The demonſtration is contained in that of the laſt problem.

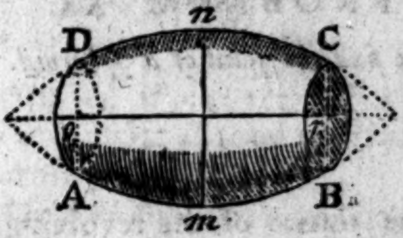
The ſolidity of the ſegment cannot be found independent of the length of the ſpindle. The rule may be ſeen in the Corollary to the laſt problem.

4. From the ſquare of half the length of the ſpindle take $\frac{1}{3}$ of the ſquare of half the length of the fruſtum, and multiply the remainder into the ſaid half length.

5. From this product take that of the generating area and central diſtance, and the remainder multiplied by 6.2832 will give the content of the fruſtum.

EXAMPLES.

1. What is the ſolidity of the fruſtum ABCD, whoſe middle diameter *nm* is 36, the diameter DA or CB 16, and the length *or* 40?



Here $\frac{1}{2} \times (20^2 \div \frac{36-16}{2}) + \frac{36-16}{4} = \frac{1}{2} \times (400 \div 10) + 5 = \frac{1}{2} \times 40 + 5 = 20 + 5 = 25 =$ *radius of the circle.*

Conſequently $25 - \frac{1}{2}\, nm = 25 - 18 = 7 =$ *central diſtance.*

And $\frac{10}{50} = \frac{1}{5} = .2 =$ *tab. verſed ſine*; *and* $.111823 =$ *tab. ſegment.*

Alſo $.111823 \times 50^2 = .111823 \times 2500 = 279.5575 =$ *area of the ſegment* D*n*C.

And $279.5575 + 320 = 599.5575 =$ *generating area o*D*n*C*r*.

Again $\surd(25^2-7^2)=\surd(625-49)=\surd576=24$ $=\frac{1}{2}$ *length of the ſpindle.*

And $(24^2-\frac{20^2}{3})\times20-599{\cdot}5575\times7)\times6.2832=$ $(556-133.3)\times20-4196.9025)\times6.2832=$ $(8853.334-4196.9025)\times6.2832=4656.4135\times$ $6.2832=29257.2904$ *ſolidity required.*

2. The middle diameter of the fruſtum of a circular ſpindle is 32, the diameter at the end 24, and the length 40: what is the ſolidity?

Anſ. 27287.5411256 *cubic inches.*

PROBLEM XX.

To find the ſolid of a ſpheroid.

RULE.*

Multiply the ſquare of the revolving axe by the fixed axe, and this product again by .5236, and it will give the ſolidity required.

Where note that .5236 is $=\frac{1}{6}$ of 3.14159.

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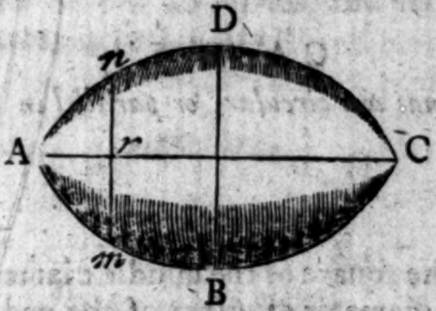
* *Demon.* Let AC $=a$, DB $=b$, A$r=x$, $rn=y$, and $p=3{\cdot}14159$, &c.

Then $a^2 : b^2 :: x\times(a-x) : \frac{b^2}{a^2}\times(ax-x^2)=y^2$ by the property of the ellipſis.

And therefore the fluxion of the ſolid $(=py^2\dot{x})=\frac{pb}{a}$ $(ax\dot{x}-x^2\dot{x})$; and its fluent $\times\frac{pb^2}{a^2}\times(\frac{1}{2}ax^2-\frac{1}{3}x^3)=$ ſegment

EXAMPLES.

1. In the prolate ſpheroid ABCD, the tranſverſe or fixed axe AC is 90, and the conjugate or revolving axe DB is 70: what is the ſolidity?



Here $DB^2 \times AC \times .5237 = 70^2 \times 90 \times .5236 = 4900 \times 90 \times .5236 = 441000 \times .5236 = 230907.6 =$ *ſolidity required.*

2. What is the ſolidity of a prolate ſpheroid, whoſe fixed axe is 100, and its revolving axe 60.

Anſ. 188496.

3. What is the ſolidity of an oblate ſpheroid, whoſe fixed axe is 60, and its revolving axe 100?

Anſ. 314160.

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nAm. Which, when $x = a$, becomes $\frac{pb^2}{a^2} \times (\frac{1}{2}a^3 - \frac{1}{3}a^3) = \frac{pab^2}{6} =$ content of the whole ſpheroid. Q. E. D.

If f be put = fixed axe, r = revolving axe, $q = (f^2 \backsim r^2) \div f^2$, and $p = 3.1415$, &c.

Then will $prf\sqrt{1 + \frac{1}{3}q} =$ ſurface of the oblate ſpheroid, and $prf\sqrt{1 - \frac{1}{3}q} =$ that of the prolate ſpheroid.

PRO-

PROBLEM XXI.

To find the content of the middle fruſtum of a ſpheroid, its length, the middle diameter, and that of either of the ends being given.

CASE I.

When the ends are circular, or parallel in the revolving axis.

RULE.*

To twice the ſquare of the middle diameter add the ſquare of the diameter of either of the ends, and this ſum multiplied by the length of the fruſtum, and the product again by .2618, will give the ſolidity.

Where note that $.2618 = \frac{1}{12}$ of 3.14159.

EXAM-

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* *Demon.* Let $\text{A}d = a$, $\text{DO} = b$, $\text{EN} = b$, $no = c$, $ro = x$, $re = y$, and $p = 3.14159$, &c.

Then $a : b^2 :: a^2 - x^2 : \frac{b^2}{a^2} \times (a^2 - x^2) = b^2 - \frac{b^2x^2}{a^2} = y^2$ by the property of the ellipſis.

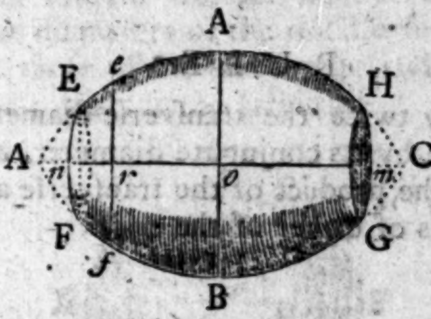
And alſo $a^2 : b^2 :: \overline{a^2 - c^2} : \frac{b^2 \times (a^2 - c^2)}{a^2} = b^2$; or $a^2 = b^2 c^2 \div (b^2 - b^2.)$

Whence, by ſubſtituting this value of a^2 in the former equation, we ſhall have $y^2 = b^2 - \frac{b4x^2 - b^2b^2x^2}{b^2 c^2} = b^2 - \frac{b^2x^2 - b^2x^2}{c^2} = b^2 - \frac{x^2}{c^2} \times (b^2 - b^2)$.

And

EXAMPLES.

1. In the middle fruſtum of a ſpheroid EFGH, the middle diameter DB is 50 inches, and that of either of the ends EF or GH 40 inches, and its length nm 18 inches: what is its ſolidity?



Here $(50^2 \times 2 + 40^2) \times 18 \times .2618 = (2500 \times 2 + 1600) \times 18 \times .2618 = (5000 + 1600) \times 18 \times .2618 = 6600 \times 18 \times .2618 = 118800 \times .2618 = 31101.84$ *cubic inches, the anſwer.*

2. What is the ſolidity of the middle fruſtum of a prolate ſpheroid, the middle diameter being 60, that of either of the two ends 36, and the diſtance of the ends 80? *Anſ.* 177940.224.

And conſequently the fluxion of the ſolid $(py^2\dot{x}) = pb^2\dot{x} - \frac{px^2\dot{x}}{c^2} \times (b^2 - b^2)$; the fluent of which is $= pb^2x - \frac{px^3}{3c^2} \times (b^2 - b^2)$; which, when $x = c$, becomes $pb^2c - \frac{pcb^2 - pcb^2}{3} = \frac{pc \times (2b^2 + b^2)}{3} = \frac{pc}{12} \times \overline{8b^2 + 4b^2}$. Q.E.D.

3. What is the ſolidity of the middle fruſtum of an oblate ſpheroid, the middle diameter being 100, that of either of the ends 80, and the diſtance of the ends 36?

Anſ. 248814.72.

CASE II.

When the ends are elliptical or perpendicular to the revolving axis.

RULE.*

1. Multiply twice the tranſverſe diameter of the middle ſection by its conjugate diameter, and to this product add the product of the tranſverſe and conjugate diameters of either of the ends.

2. Mul-

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* *Demon.* Put $SO = a$, $EO = b$, $om = r$, $on = x$, $An = y$, $nc = z$, and $p = 3.14159$, &c.

Then $a^2 : b^2 :: a^2 - x^2 : \frac{b^2}{a^2} \times (a^2 - x^2) = y^2$ by the property of the ellipſis.

And, ſince AcD is an ellipſis ſimilar to EmF, it will be $b : r :: y : \frac{ry}{b} = z$; as is ſhewn by the writers on Conics.

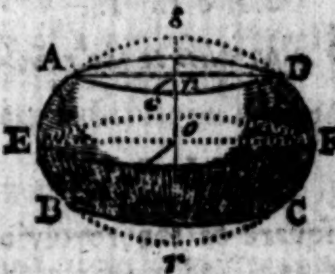
But the fluxion of the ſolid AEFD is $pyz\dot{x} = py\dot{x} \times \frac{ry}{b} = \frac{pry^2\dot{x}}{b} = \frac{pr\dot{x}}{b} \times \frac{b^2 \times (a^2 - x^2)}{a^2} = prb\dot{x} \times \frac{a^2 - x^2}{a^2}$. And the fluent $= prbx \times \frac{a^2 - \frac{1}{3}x^2}{a^2}$. Which, by ſubſtituting for a^2 its value $\frac{b^2 x^2}{b^2 - y^2}$, becomes $= prx \times \frac{2b^2 + y^2}{3b} = px \times \frac{2}{3}rb + \frac{ry^2}{3b}$.

And

2. Multiply the ſum, thus found, by the diſtance of the ends, or the height of the fruſtum, and the product again by .2618, and it will give the ſolidity required.

EXAMPLES.

1. In the middle fruſtum ABCD of an oblate ſpheroid, the diameters of the middle-ſection EF are 50 and 30; thoſe of the end AD 40 and 24; and its height *ne* 18; what is the ſolidity?



Here $(50 \times 2 \times 30 + 40 \times 24) \times 18 \times .2618 = (3000 + 960) \times 18 \times .2618 = 3960 \times 18 \times .2618 = 71280 \times .2618 = 18661.104 =$ *ſolidity required.*

2. In the middle fruſtum of a prolate ſpheroid, the diameters of the middle ſection are 100 and 60; thoſe of the end 80 and 48; and the length 36: what is the ſolidity? *Anſ.* 149288.832.

And this again, by putting z for its equal $\frac{ry}{b}$, becomes $=$ $\frac{px}{3} \times \overline{2rb + yz} =$ fruſtum EFDA. Or $\frac{p \times ne}{12} \times (2EF \times 2om + AD \times 2nc) =$ middle fruſtum ABCD. Q. E. D.

3. In the middle fruſtum of an oblate ſpheroid, the diameters of the middle ſection are 100 and 60: thoſe of the end 60 and 36; and the length 80: what is the ſolidity of the fruſtum? *Anſ.* 296567.04.

PROBLEM XXII.

To find the ſolidity of the ſegment of a ſpheroid.

CASE I.

When the baſe is parallel to the revolving axis.

RULE.*

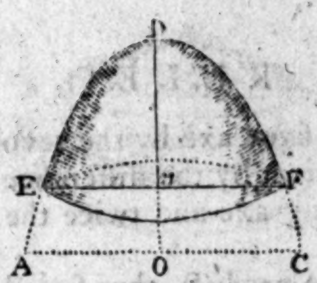
1. Divide the ſquare of the revolving axis by the ſquare of the fixed axe, and multiply the quotient by the difference between three times the fixed axe and twice the height of the ſegment.

2. Multiply the product, thus found, by the ſquare of the height of the ſegment, and this product again by .5236, and it will give the ſolidity required.

EXAMPLES.

1. In the prolate ſpheroid DEFD, the tranſverſe axis 2DO is 100, the conjugate AC 60, and the height D*n* of the ſegment EBF 10: what is the ſolidity?

*** This rule is formed from the theorem for the ſegment in the demonſtration to problem the 20th.**



Here $(\frac{60^2}{100^2} \times \overline{300-10}) \times 10^2 \times .5236 = .36 \times 280 \times 10^2 \times .5236 = 100.80 \times 100 + .5236 = 10080 \times .5236 = 5277.888 =$ *ſolidity required.*

2. The axes of a prolate ſpheroid are 50 and 30: what is the ſolidity of that ſegment whoſe height is 5, and its baſe perpendicular to the fixed axe?

Anſ. 659.746.

3. The diameters of an oblate ſpheroid are 100 and 60; what is the ſolidity of that ſegment whoſe height is 12, and its baſe perpendicular to the conjugate axe? *Anſ.* 32672.64.

CASE II.

When the baſe is perpendicular to the revolving axis.

The content of the ſegment may alſo be found by the following theorem:

$(D^2 + 4d^2) \times \frac{1}{6} n b =$ content of the ſegment; D being the diameter of the baſe, $d =$ diameter in the middle, $b =$ height, and $n = .7854 =$ area of a circle whoſe diameter is 1.

RULE.

RULE.*

1. Divide the fixed axe by the revolving axe, and multiply the quotient by the difference between three times the revolving axe and twice the height of the ſegment.

2. Multiply the product, thus found, by the ſquare of the height of the ſegment, and this product again by .5236, and it will give the ſolidity required.

EXAMPLES.

1. In the oblate ſpheroid $aEbF$, the tranſverſe axe EF is 100, the conjugate ab 60, and the height an, of the ſegment aAD, 12: what is the ſolidity?

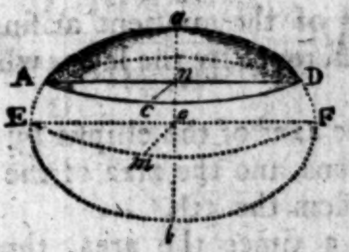
---

* *Demon.* Put $ao = a$, $Eo = b$, $om = r$, $an = x$, $An = y$, $nc = z$, and $p = 3.14159$, &c. Then will $a^2 : b^2 :: a^2 - (a-x)^2$ or $(2ax - x^2) : \frac{b^2 \times (2ax - x^2)}{a^2} = y^2$ by the property of the ellipſe.

And, ſince ACD is an ellipſe ſimilar to EmF, it will be $b : r :: y : \frac{ry}{b} = z$; as is ſhewn by the writers on Conics.

But the fluxion of the ſolid $aACD = pyz\dot{x} = py\dot{x} \times \frac{ry}{b} = \frac{pry^2\dot{x}}{b} = \frac{pr\dot{x}}{b} \times \frac{b^2 \times (\overline{2ax - x2})}{a^2}$; whoſe fluent is $= \frac{prb}{a}x^2 - \frac{prb}{3a^2}x^3$; which, when $x = b =$ the height of the ſegment, becomes $(3ab - b^3) \times \frac{prb}{3a^2}$. Whence, ſince $r = a$, we ſhall have $(3ab^2 - b^3) \times \frac{pb}{3a} =$ ſolidity of the ſegment.

Q. E. D.



Here 156 (=*dif. of 3ab and 2an*) × $1\frac{2}{3}$ (= EF, *or* $\frac{100}{ab}$ × 144 (=*ſquare of an*) × .5236 = $\frac{156 \times 5}{3}$ × 144 × .5236 = 52 × 5 × 144 × .5236 = 260 × 144 × 5236 = 37440 × .5236 = 19603.584 = *ſolidity required.*

2. Required the content of the ſegment of a prolate ſpheroid; its height being 6, and the axes 40 and 24. *Anſ.* 2450.44226.

PROBLEM XXIII.

To find the ſolidity of an elliptic ſpindle.

RULE.*

1. From three times the ſquare of the middle diameter take 4 times the ſquare of the diameter between the middle and the end; and from 4 times this

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* *Demon.* Let $Fv = a$, $ov = b$, $\frac{1}{2}$ the tranſverſe diameter $= c$, $Do = d$, $va = x$, and vc or $an = y$.

Then, by the property of the ellipſe, $c : d :: \sqrt{c^2 - x^2} : \frac{d}{c}\sqrt{c^2 - x^2} = co$; hence cv or $an = co - ov = \frac{d\sqrt{c^2 - x^2}}{c} - b = y$, and the fluxion of the ſolid $= py^2\dot{x} = p\dot{x} \times$

(a²

this laſt diameter take 3 times the ſaid middle diameter; and $\frac{1}{4}$ of the quotient ariſing from dividing the former difference by the latter will give the *central diſtance.*

2. Find the axes of the ellipſis by problem the 2d, in conic ſections, and the area of the generating ſegment by problem the 5th.

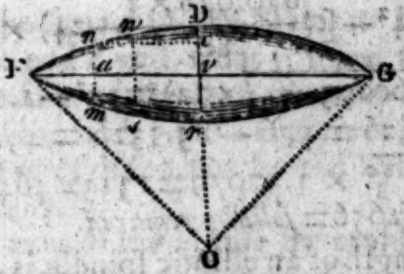
3. Divide 3 times the area, thus found, by the length of the ſpindle, and from the quotient ſubtract the middle diameter; then multiply the remainder by 4 times the central diſtance, and ſubtract the product from the ſquare of the middle diameter; and this difference multiplied by $\frac{1}{3}$ of the length of the ſpindle, and the product again by 1.57079 will give the ſolidity.

EXAMPLES.

1. What is the ſolidity of the elliptic ſpindle F*r*GD, whoſe length FG is 80, the middle diameter D*r* 24, and the diameter *ws* at $\frac{1}{4}$ of the length 18.99094?

Here

$(d^2 - \frac{d^2x^2}{c} - \frac{2bd\sqrt{c^2 - x^2}}{c} + b^2) = p\dot{x} \times (d^2 - b^2 - \frac{d^2x^2}{c^2} - \frac{2bd\sqrt{c^2 - x^2}}{c}) - b = pd^2\dot{x} \times \frac{a^2 - x^2}{c^2} - 2pby\dot{x}$; and its fluent $= pd^2 x \times \frac{3a^2 - x^2}{3c^2} - 2bp \times$ area *van*D $=$ fruſtum D*nmr*. And when $x = a$ the above theorem will become $\frac{2pd^2}{3c^2} \times a^3 - 2bp \times$ area FD*v* $=$ DF*r* the half of the ſpindle.



Here $(\overline{3 \times 24^2 - 18.99094^2 \times 4} \div \overline{18.99094 \times 4 - 3 \times 4})$ $\times \frac{1}{4} = (\overline{1728 - 1442.62320833} \div \overline{75.96376 - 72}) \times \frac{1}{4}$ $= (285.37679167 \div 3.96376) \times \frac{1}{4} = 71.996 \times \frac{1}{4} =$ $17.999 = 18$ *nearly, for the central diſtance* Ov.

And $18 + \frac{24}{2} = 18 + 12 = 30\frac{1}{2}$ *conjugate* DO. *And* $\frac{30}{2} \times 30 \div \sqrt{30^2 - 18^2} = 40 \times 30 \div \sqrt{900 - 324} =$ $1200 \div \sqrt{576} = \frac{1200}{24} = 50 = \frac{1}{2}$ *tranſverſe.*

Alſo $\frac{12}{2 \times 30} = \frac{12}{60} = \frac{1}{5} = .2 =$ *tab. verſed ſine; the circular ſegment to which is* $.111823 \therefore .111823 \times 100$ (AB) $\times 60$ (DF) $= 11.1823 \times 60 = 670.938 =$ *generating ſegment* FDG.

Whence

ſpindle. And if from the ſemi-ſpindle there be taken the fruſtum, there will remain $pd^2e^2 \times \frac{3a - e}{3c^2} - 2pb \times$ area Fna = ſegment nFm, e being the height Fa of the ſegment. But to convert theſe rules into thoſe given in the text, let Dr = D, $nm = d$, $ws = m$, $va = b$, Ov = c, the ſegment nD$c = s$, and $n = .7854$.

Then, by the property of the ellipſe, $(c + \frac{1}{2}D)^2 - (c + \frac{1}{2}d) : 4 :: (c + \frac{1}{2}D)^2 - (c + \frac{1}{2}m)^2 : 1$; hence $4 \times (c + \frac{1}{2}m)^2 - 3 \times (c + \frac{1}{2}D)^2 = (c + \frac{1}{2}d)^2$, or $c = \frac{1}{4} \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}$.

And

Whence $24^2 - [(\frac{670.938 \times 3}{80} - 24) \times 72$ *or* $40v] =$ $24^2 - (\overline{25.160175 - 24} \times 72) = 24^2 - 1.160175 \times 72 =$ $24^2 - 83.5326 = 576 - 83.5326 = 492.4674$. *And* $\overline{492.4674} \times \frac{80}{3} \times 1.57079 = 13132.464 \times 1.57079 =$ $20628.34312656 =$ *ſolidity required.*

2. The length of an elliptic ſpindle is 40, the middle diameter 12, and the diameter at ¼ of the length 9.49546: what is the ſolidity? *Anſ.* 2578.56.

PROBLEM XXIV.

To find the ſolidity of the middle fruſtum of an elliptic ſpindle; the length, the diameters of the middle and end, and another parallel thereto at ¼ of the length of the fruſtum, being given.

RULE.*

1. From the ſum of 3 times the ſquare of the middle diameter and the ſquare of that of the end

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And the laſt two theorems become $\frac{1}{3} nb \times (\frac{1}{2} D^2 + d^2 - 2) \times \frac{3D^2 + d^2 - 4m}{-3D - d + 4m} \times (-D + d + \frac{3s}{b})$ for the fruſtum D n m r; and $\frac{1}{3} nl \times (2D^2 - 2 \times \frac{2D^2 - 4m^2}{-3D + 4m} \times -D + \frac{3s}{l})$ for the ſemi-ſpindle D F r, where $l = v$ F and s = area of F D v.

Q. E. D.

[*] The demonſtration of this rule is contained in that of the laſt problem.

The ſolidity of a ſegment of an elliptic ſpindle cannot be found independently of the axes of the ſpheroid.

take 4 times the ſquare of the diameter between the middle and the end; and from 4 times the laſt diameter take the ſum of the leaſt diameter and 3 times that of the middle, and ¼ of the quotient ariſing from dividing the former difference by the latter will give the *central diſtance.*

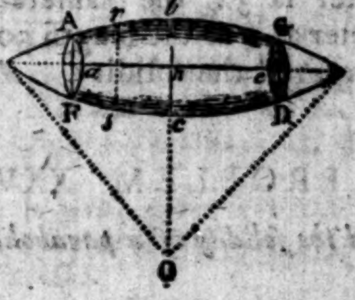
2. Find the axes of the ellipſe by problem the 2d, and the area of the elliptical ſegment whoſe chord is the length of the fruſtum by problem the 5th.

3. Divide three times the area thus found by the length of the fruſtum, and from the quotient ſubtract the difference between the middle diameter and that of the end, and multiply the remainder by 8 times the central diſtance.

4. Then from the ſum of the ſquare of the leaſt diameter, and twice the ſquare of that in the middle, take the product laſt found, and this difference multiplied by the length, and the product again by .261799, &c. will give the ſolidity required.

EXAMPLES.

1. What is the ſolidity of the fruſtum AFDG, whoſe length *ae* is 28, the middle diameter *bc* 24, the diameter AF of the end 21.6, and that *rs* at ¼ of the length 23.409?



Here $\frac{24^2\times3+(21.6)^2-4\times(23.409)^2}{4\times23.409-(21.6+3\times24)}=\frac{2194.56-2191.925124}{93.63636-93.6}=\frac{2194.56-2191.925124}{.03636}=\frac{2.634876}{.03636}=72.46.$

And $\frac{72.46}{4}=18.11=18$ *nearly* $=$ *central diſtance on.*

Alſo $\frac{30\,(=bo)\times14\,(=an)}{\sqrt{30^2(=bo^2)-28.8^2(=om^2)}}=\frac{420}{\sqrt{900-829.44}}=\frac{420}{\sqrt{70.56}}=\frac{420}{8.4}=50=\frac{1}{2}$ *tranſverſe diameter.*

And $\frac{1.2\,(=bm)}{60\,(=bp)}=.02=$ *tab. verſed ſine; and* $.003748=$ *circular ſegment belonging to* $.02$; $\therefore .003748\times100\,(=\text{LN})\times60\,(=bp)=22.488=$ *elliptic ſegment, of which* A*b*G *is the arc.*

Whence $24^2\times2+(21.6)^2-\left(\overline{\frac{3\times22.488}{28}-2.4\,(=\text{diff. of } bc \text{ and AF})}\right)\times8\times18=1152+466.56-(\overline{2.4094-2.4}\times144)=1618.56-.0094\times144=1618.56-1.3536=1617.2064.$

And $1617.2064\times28\,(=ae)\times.261799=45281.779^2\times.261799=11854.7245127808=$ *ſolidity required.*

2. In the middle fruſtum of an elliptic ſpindle, the middle diameter is 32, the diameter at the end 24, and the diameter at $\frac{1}{4}$ of the length 30.15756, and the length 40: required the ſolidity. *Anſ.* 27419.8219.

PROBLEM XXV.

To find the ſolidity of a parabolic conoid.

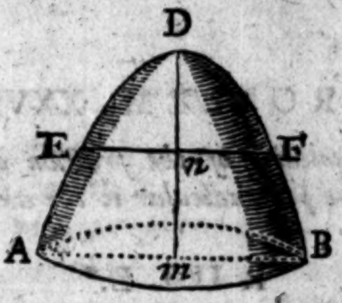
RULE.

RULE.*

Multiply the area of the baſe by half the altitude, and the product will be the content.

EXAMPLES.

1. What is the ſolidity of the paraboloid ADB, whoſe height D*m* is 84, and the diameter BA of its circular baſe 48?



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* *Demon.* Let D*m* $= a$, B*m* $= b$, D*n* $= x$, E*n* $= y$, and $p = 3.14159$, &c.

Then by the nature of the parabola $a : b^2 :: x : y^2$, or $\frac{b^2 x}{a} = y^2$; wherefore $\frac{pb^2 x\dot{x}}{a}$ $(= py^2\dot{x})$ $=$ the fluxion of the ſolid, and $\frac{pb^2 x^2}{2a}$ $=$ its fluent; which, when x becomes $= a$, is $\frac{1}{2}pab^2$ for the whole ſolid, or for any ſegment whoſe height is $= a$, and the radius of its baſe $= b$. Q. E. D.

Coroll. The parabolic conoid is $= \frac{1}{2}$ its circumſcribing cylinder.

Note. The rule given above will hold for any ſegment of the paraboloid, whether the baſe be perpendicular or oblique to the axe of the ſolid.

Here $8^2 \times .7854 \times 42 \; (= \frac{1}{2}\text{D}m) = 2304 \times .7854 \times 42 = 1809.5616 \times 42 = 76001.5872 =$ *ſolidity required.*

2. What is the ſolidity of a paraboloid, whoſe height is 60, and the diameter of its circular baſe 100? *Anſ.* 235620.

3. Required the ſolidity of a parabolic conoid, whoſe height is 30, and the diameter of its baſe 40? *Anſ.* 18849.6.

4. Required the ſolidity of a parabolic conoid, whoſe height is 50, and the diameter of its baſe 100? *Anſ.* 126350.

PROBLEM XXVI.

To find the ſolidity of the fruſtum of a paraboloid, when its ends are perpendicular to the axe of the ſolid.

RULE.*

Multiply the ſum of the ſquares of the diameters of the two ends by the height of the fruſtum, and the product again by .3927, and it will give the ſolidity.

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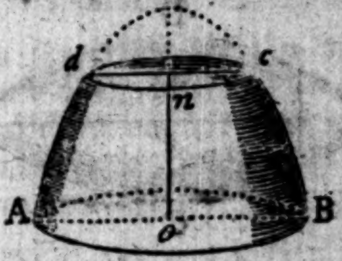
[*] *Demon.* The ſegment whoſe baſe is B, and altitude A, is $= \frac{1}{2}\text{AB}$, and that whoſe baſe is b and altitude a is $= \frac{1}{2}ab$, by the laſt problem: wherefore the fruſtum, or the difference of the ſegment is $\frac{1}{2}\text{AB} - \frac{1}{2}ab$. But $\text{B} - b : \text{A} - a \; (d) :: \text{B} : \text{A} = \frac{\text{B}d}{\text{B} - b}$; and $\text{B} - b : d :: b : a = \frac{bd}{\text{B} - b}$, by the nature of the paraboloid; and theſe values of A and a being ſubſtituted for them will make $\frac{1}{2}\text{AB} - \frac{1}{2}ab = \frac{d\text{B} - db}{2\text{B} - 2b} = \frac{1}{2}d \times (\text{B} + b)$ which is the ſame as the rule. Q. E. D.

EXAMPLES.

1. Required the ſolidity of the parabolic fruſtum $ABcd$, the diameter AB of the greater end being 58, that of the leſſer end dc 30, and the height no 18.



Here $(58^2+30^2)\times 18\times .3927=(3364+900)\times 18\times .3927=4264\times 18\times .3927=76752\times .3927=30140.5104=$ *ſolidity required.*

2. What is the ſolidity of the fruſtum of a parabolic conoid, the diameter of the greater end being 60, that of the leſſer end 48, and the diſtance of the ends 18? *Anſ.* 41733.0144.

PROBLEM XXVII.

To find the ſolidity of a parabolic ſpindle.

RULE.*

Multiply the ſquare of the middle diameter by the length of the ſpindle, and the product again by .418879, and it will give the ſolidity.

EXAM-

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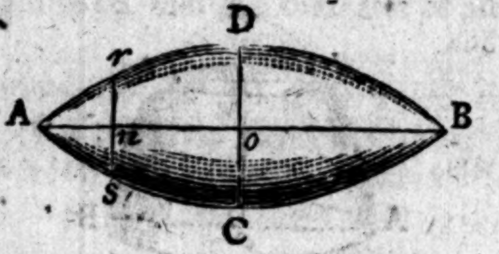
* *Demon.* Put $Do=a$, $Ao=b$, $c=3.14159$, &c. $An=x$, and $rn=y$.

EXAMPLES.

1. The length of the parabolic ſpindle A C B D is 60, and the middle diameter D C 34: what is the ſolidity?



Here $34^2 \times 60 \times .418879 = 1156 \times 60 \times .418879 = 69360 \times .418879 = 29053.44744 =$ *ſolidity required.*

2. The length of a parabolic ſpindle is 9 feet, and the middle diameter 3 feet: what is the ſolidity?

Anſ. 33.929199.

Then it will be, by the property of the parabola, $b^2 : a :: 2bx - x^2$ (A$n \times n$B) $: a \times \frac{2bx - x^2}{b^2} = rn = y$; which, by putting $p = \frac{b^2}{a}$ ($=$ parameter of Do,) becomes $\frac{2bx - x^2}{p}$; and hence the fluxion of the ſolid $= cy^2\dot{x} = \frac{c\dot{x}}{p^2} \times \overline{2bx - x}|^2$; the fluent of which is $cx^3 \times \frac{\frac{4}{3}b^2 - bx + \frac{1}{5}x^2}{p^2} =$ general expreſſion for the ſegment $r\rho s$; and therefore when $x = b$, we ſhall have $\frac{8cb5}{15p^2} = \frac{8}{15}ca^2b =$ ſemi-ſpindle D A C, and $a^2b \times .418879 =$ whole ſpindle B D A C. Q. E. D.

PRO-

PROBLEM XXVIII.

To find the ſolidity of the middle fruſtum of a parabolic ſpindle.

RULE.*

Add 8 times the ſquare of the middle diameter, 3 times the ſquare of the leſs, and 4 times the product of thoſe diameters into one ſum; then this ſum being multiplied by the length, and the product again by .05236 will give ſolidity.

EXAMPLES.

1. In the middle fruſtum EFGH, of the parabolic ſpindle ACBD, the middle diameter DC is 36, the

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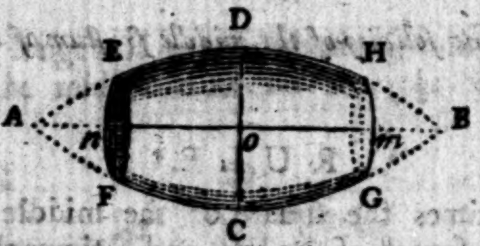
* *Demon.* Let $ox = a - x = z$, and the other letters as in the laſt problem.

Then if from the value of the ſemi-ſpindle $\frac{8cb5}{15p^2}$ there be taken $cx3 \times \frac{\frac{4}{3}b^2 - bx + \frac{1}{5}x^2}{p^2}$ the value of the ſegment EAF, there will remain $\frac{cz}{p^2} \times \overline{b4 - \frac{2}{3}b^2z^2 + \frac{1}{5}z4} = \frac{ca^2z}{b4} \times (b4 - \frac{2}{3}b^2z^2 \times \frac{1}{5}z4) =$ the value of the fruſtum DEFC; and if inſtead of z there be ſubſtituted its value $b \surd \frac{a-y}{a}$ the value of the ſaid fruſtum will be denoted by $c \times no \times \frac{8DC^2 + 4no \times EF + 3EF^2}{60}$, and conſequently $nm \times .05236 \times 8DC^2 + 4no \times EF + 3EF^2 =$ whole fruſtum. Q. E. D.

diameter of the end EF is 20, and the length *nm* 36: what is the ſolidity?



Here $(36^2 \times 8 + 20^2 \times 3 + 4 \times 36 \times 20) \times 36 \times .05236 = (10368 + 1200 + 2880) \times 36 \times .05236 = 14448 \times 36 \times .05236 = 520128 \times .0.5236 = 27233.90228 =$ *ſolidity required.*

2. Required the ſolidity of the middle fruſtum of a parabolic ſpindle, the middle diameter being 32, the diameter at the end 24, and the length 40?

Anſ. 17210.448.

PROBLEM XXIX.

To find the ſolidity of an hyperboloid.

RULE.[*]

To the ſquare of the radius of the baſe add the ſquare of the middle diameter between the baſe and the

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[*] *Demon.* Let t = tranſverſe, and c = conjugate diameter of the generating hyperbola, $p = 3.14159$, y, Y, the ordinates, or ſemi-diameters of the ends of any fruſtum of the hyperboloid, x = its altitude, and A = diſtance of the leſs ordinate y from the vertex of the whole ſolid.

Then

the vertex; and this ſum multiplied by the altitude, and the product again by .5236 will give the ſolidity.

Then ſince $Y^2 = \frac{(t+A+x)\times(A+x)}{t^2} \times c^2$, we ſhall have the fluxion of the ſolid $= pY^2\dot{x} = pc^2\dot{x} \times \frac{At + A^2 + 2Ax + tx + x^2}{t^2}$, and its fluent $= pc^2x \times \frac{At + A^2 + Ax + \frac{1}{2}tx + \frac{1}{3}x^2}{t^2}$; and this, by ſubſtituting $\frac{y^2}{c^2}$ for $\frac{At + A^2}{t^2}$, and $\frac{Y^2}{c^2}$ for $\frac{At + A^2 + 2Ax + tx + x^2}{t^2}$ becomes $(Y^2 + y^2 - \frac{c^2x^2}{3t^2}) \times \frac{1}{2}px =$ ſolidity of the fruſtum.

But to convert this into the rules given in the text, let D, δ, d, be the greateſt, middle, and leaſt diameters, $x =$ abſciſſa whoſe ordinate is δ, and $a =$ altitude. Then we ſhall have theſe three equations:

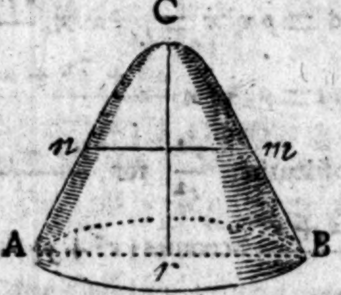
$$t^2\delta^2 = c^2 \times \overline{t+x} \times x$$
$$t^2d^2 = c^2 \times \overline{t+x-\tfrac{1}{2}a} \times \overline{x-\tfrac{1}{2}a}$$
$$t^2D^2 = c^2 \times \overline{t+x+\tfrac{1}{2}a} \times \overline{x+\tfrac{1}{2}a}$$

From the ſum of the two latter of which ſubtract the double of the former, and there will reſult $t^2 \times \overline{D^2 - 2\delta^2 + d^2} = \frac{1}{2}a^2c^2$; and hence $\frac{a^2c^2}{3t^2} = \frac{2D^2 - 4\delta^2 + 2d}{3}$. Which being ſubſtituted for it in the theorem above will give $\frac{D^2 + 4\delta^2 + d}{6} \times ap$ for the content of the fruſtum; which is the ſame as the following rule given in the text.

And if d the leaſt diameter be ſuppoſed to become infinitely little, or nothing, the rule will become $\frac{D^2 + 4\delta^2}{6} \times ap = \overline{D^2 + 4\delta^2} \times a \times .5236$. Q. E. D.

EXAMPLES.

1. In the hyperboloid ACB, the altitude Cr is 10, the radius Ar of the baſe 12, and the middle diameter nm 15.8745: what is the ſolidity?



Here $\overline{15.8745^2+12^2}\times10\times.5236=\overline{251.99975+144}\times10\times.5236=395.99975\times10\times.5236=3959.9975\times.5236=2073.4558691=$ *ſolidity required.*

2. In an hyperboloid the altitude is 50, the radius of the baſe 52, and the middle diameter 68; what is the ſolidity? *Anſ.* 191847.

PROBLEM XXX.

To find the ſolidity of the fruſtum of an hyperbolic conoid.

RULE.*

Add together the ſquares of the greateſt and leaſt ſemi-diameters, and the ſquare of the whole diameter in

---

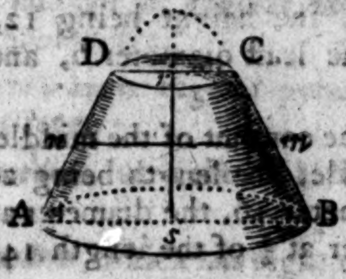
[*] The demonſtration of this rule is contained in that of the laſt problem.

The

in the middle, then this ſum being multiplied by the altitude, and the product again by .52359 will give the ſolidity.

EXAMPLES.

1. In the hyperbolic fruſtum ADCB, the length rs, is 20, the diameter AB of the greater end 32, that DC of the leſſer end 24, and the middle diameter nm, 28.1708: required the ſolidity.



The rule for the hyperbolic ſpindle is the ſame as that for the elliptic ſpindle, page 175.

Or, if D = middle diameter, m = that at $\frac{1}{4}$ of the length, S = generating area of the hyperbola, L = length of the ſpindle, and $p = 3.14159$, &c.

Then will $\left(D + \frac{4m^2 - 3D^2}{4m - 3D} \times \frac{3S - D}{L}\right) \times \frac{1}{6} pL$ = ſolidity of the ſpindle. And if the generating hyperbola be equilateral, then will $\left(3S \times \frac{L^2 + D^2}{LD} - L^2\right) \times \frac{1}{6} pL$ = ſolidity of the ſpindle.

And, if l = length of the fruſtum, s = generating area, and the other letters as before; then will $\left(2D^2 + d^2 + \frac{4m^2 - 3D^2 - d^2}{4m - 3d - d} \times \overline{\frac{3s}{l} + d - D}\right) \times \frac{1}{12} pl$ = ſolidity of the middle fruſtum of an hyperbolic ſpindle.

But if the generating hyperbola be equilateral, the fruſtum will be $= \left(\frac{3}{2} d^2 - l^2 + \frac{3s}{l} \times \frac{l^2 + D^2 - d^2}{D - d}\right) \times \frac{1}{6} pl$.

Here $(16^2+12^2+28.1708^2) \times 20 \times .52359 = (256 + 144 + 793.5939) \times 20 \times .5239 = 1193.5939 \times 20 \times .5239 = 23871.878 \times .5239 = 12499.07660202 =$ *ſolidity required.*

2. What is the ſolidity of the fruſtum of any hyperbolic conoid, whoſe greater diameter is 96, leſſer diameter 54, the middle diameter 76.4264392, and the altitude 12.5? *Anſ.* 116160.66.

3. Required the ſolidity of the fruſtum of an hyperbolic conoid, the height being 12, the greateſt diameter 10, the leaſt diameter 6, and the middle diameter $8\frac{1}{2}$? *Anſ.* 667.59.

4. What is the content of the middle fruſtum of an hyperbolic ſpindle, the length being 20, the middle or greateſt diameter 16, the diameter at each end 12, and the diameter at $\frac{1}{4}$ of the length $14\frac{1}{2}$? *Anſ.* 3248.938.

5. Required the content of the ſegment of any ſpindle, its length being 10, the greateſt diameter 8, and the middle diameter 6? *Anſ.* 272.272.

Note. The content of any ſpindle formed by the revolution of a conic ſection about its axis may be found by the following rule:

Add together the ſquares of the greateſt and leaſt diameters, and the ſquare of double the diameter in the middle between the two, and this ſum multiplied by the length and the product again by .1309 will give the ſolidity.

And the rule will never deviate much from the truth when the figure revolves about any other line which is not the axis.

OF THE

REGULAR BODIES.

A REGULAR BODY is a ſolid contained under a certain number of ſimilar and equal plane figures.

The whole number of regular bodies which can poſſibly be formed is five.

1. The *Tetraedron*, or regular pyramid, which has four triangular faces.

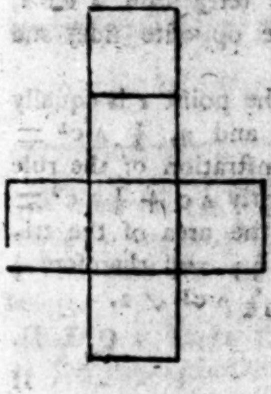
2. The *Hexaedron*, or cube, which has ſix ſquare faces.

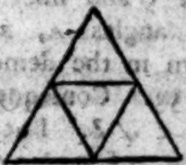
3. The *Octaedron*, which has eight triangular faces.

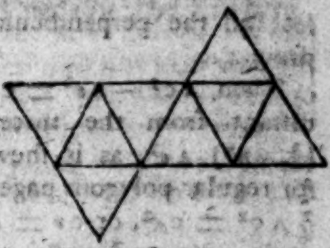
4. The *Dodecaedron*, which has twelve pentagonal faces.

5. The *Icoſaedron*, which has twenty triangular faces.

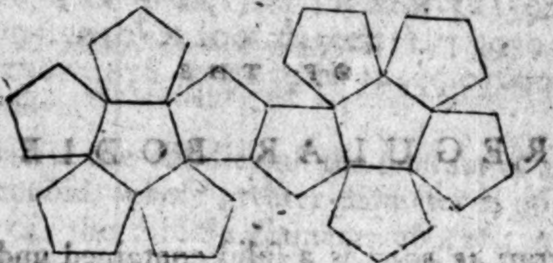
If the following figures are made of paſteboard, and the lines be cut half through, ſo that the parts may be turned up and glued together, they will repreſent the five regular bodies here mentioned.

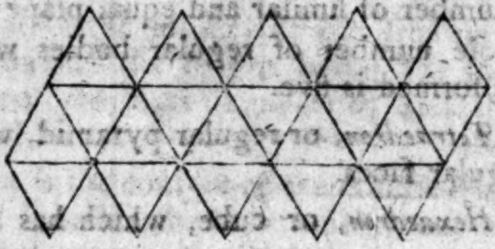






PRO-





PROBLEM I.

To find the ſolidity of a tetraedron.

RULE.*

Multiply $\frac{1}{12}$ of the cube of the linear ſide by the ſquare root of 2, and the product will be the ſolidity.

EXAM-

---

* *Demon.* From one angle C of the tetraedron $ABCn$, let fall the perpendicular Ce, upon the oppoſite ſide, and draw Ae.

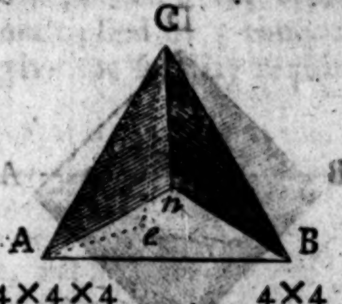
Then $AC^2 - Ae^2 = Ce^2$; and ſince the point e is equally diſtant from the three angles A, B and n, $\frac{1}{3}AC^2 = (\frac{1}{3}AB^2)\ Ae^2$, as is ſhewn in the demonſtration of the rule for regular polygons page 53. Conſequently $AC + \frac{1}{3}AC^2 = \frac{2}{3}AC^2 = Ce^2$, or $Ce = AC\sqrt{\frac{2}{3}}$. But the area of the triangle $AnB = \frac{1}{4}AB^2\sqrt{3} = \frac{1}{4}AC^2\sqrt{3}$; and therefore $\frac{1}{3}AC\sqrt{\frac{2}{3}}(\frac{1}{3}Ce) \times \frac{1}{4}AC^2\sqrt{3}(\triangle AnB) = \frac{1}{12}AC^3\sqrt{2}$.

Q. E. D.

If

EXAMPLES.

1. The linear ſide of the tetraedron ABC*n* is 4: what is the ſolidity?



$$\frac{4^3}{12} \times \sqrt{2} = \frac{4 \times 4 \times 4}{12} \times \sqrt{2} = \frac{4 \times 4}{3} \times \sqrt{2} = \frac{16}{3} \sqrt{2} = \frac{16}{3} \times 1.414 = \frac{22.624}{3} = 7.5413$$

= *ſolidity required.*

2. Required the ſolidity of a tetraedron whoſe ſide is 6? *Anſ.* 25.452.

PROBLEM II.

To find the ſolidity of an octaedron.

RULE.*

Multiply $\frac{1}{3}$ of the cube of the linear ſide by the ſquare root of 2, and the product will be the ſolidity.

EXAM-

If L be put = length of the linear edge, then will $L^2 \sqrt{3}$ = whole *ſurface of the tetraedron.*

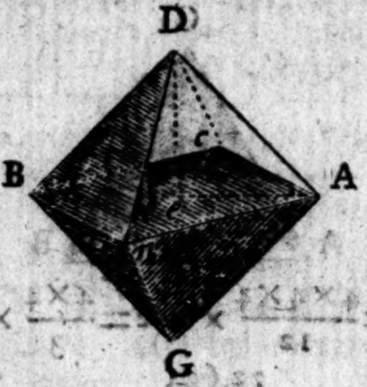
The rule for the hexaedron, or cube, has been given before.

[*] *Demon.* From the angle D of the octaedron DFGA let fall the perpendicular D*e*.

Then,

EXAMPLES.

1. What is the ſolidity of the octaedron FGAD, whoſe linear ſide is 4?



$\frac{4^3}{3} \times \sqrt{2} = \frac{64}{3} \times \sqrt{2} = 21.333$, &c. $\times \sqrt{2} =$ 21.333, &c. $\times 1.452$, &c. $= 30.16486 =$ *ſolidity required.*

2. Required the ſolidity of an octaedron whoſe ſide is 8? *Anſ.* 243.3568.

PROBLEM III.

To find the ſolidity of a dodecaedron.

Then ſince the ſolid is compoſed of two equal ſquare pyramids, each of whoſe baſes $FnAc$ are equal to the ſquare of the linear ſide AG or AD, we ſhall have $FnAc \times \frac{2}{3} De = An^2 \times \frac{2}{3} De =$ content of the ſolid.

But De evidently biſects the diagonal FA, and is equal to Fe; therefore $An^2 \times \frac{2}{3} De = An^2 \times \frac{2}{3} Fe = An^2 \times \frac{1}{3} FA = \frac{1}{3} An^2 \times \sqrt{An^2 + Ac^2} = \frac{1}{3} An^2 \sqrt{2 An^2} = \frac{1}{3} An^3 \sqrt{2}$. Q. E. D.

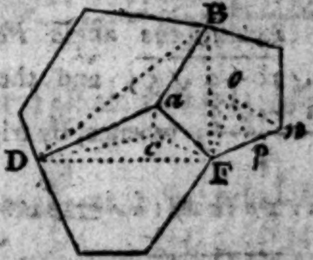
If $L =$ linear ſide as before, then will $2L^2 \sqrt{3} =$ ſurface of the octaedron.

RULE.

RULE.*

To 21 times the ſquare root of 5 add 47, and divide the ſum by 40: then the ſquare root of the quotient being multiplied by 5 times the cube of the linear ſide will give the ſolidity required.

EXAM-



* *Demon.* Let *a* be a ſolid angle of the dodecaedron, and *ac* a perpendicular falling on the equilateral plane BDF. Alſo join the points D, *c* and *c*, F.

Then the angle D*a*F contains 108 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, and the angle *a*FD contains 36 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10-2\sqrt{5}}$, the radius in both caſes being taken equal to 1.

Therefore, by Trigonometry, $\frac{1}{4}\sqrt{10-2\sqrt{5}} : \frac{1}{4}\sqrt{10+2\sqrt{5}} :: a\mathrm{D} : \mathrm{DF} = a\mathrm{D}\sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = a\mathrm{D}\frac{1+\sqrt{5}}{2} = \frac{1}{2}+\frac{1}{2}\sqrt{5}\times a\mathrm{D}$.

Again, ſince *c* is the centre of DBF, the angles *c*DF and *c*FD are each 30°, and the angle D*c*F = 120°; but the ſine of 30° is ½; and the ſine of 120° is $\frac{1}{2}\sqrt{3}$; whence, by trigonometry, $\frac{1}{2}\sqrt{3} : \mathrm{DF} :: \frac{1}{2} : \mathrm{D}c = \frac{\mathrm{DF}}{\sqrt{3}} = a\mathrm{D}\frac{1+\sqrt{5}}{2\sqrt{3}}$; and conſequently,

$$\mathrm{AC} = a\mathrm{p}^2 - \mathrm{DC}^2 = a\mathrm{D}\sqrt{\frac{3-\sqrt{5}}{6}} = a\mathrm{D}\sqrt{\frac{1}{2}-\frac{1}{6}\sqrt{5}}.$$

But a perpendicular from *a* upon the plane BDF muſt paſs through the center of the circumſcribing ſphere, and *ac* will be the verſed ſine of an arc whoſe chord is *a*D, and radius equal to that of the ſaid ſphere.

Whence $ac : a\mathrm{D} :: a\mathrm{D} : \frac{a\mathrm{D}^2}{a\mathrm{C}} = \frac{a\mathrm{D}^2}{a\mathrm{D}\sqrt{\frac{1}{2}-\frac{1}{6}\sqrt{3}}} = a\mathrm{B}$

√3

EXAMPLES.

1. The linear ſide of the dodecaedron $ABCDE$ is 3: what is the ſolidity?

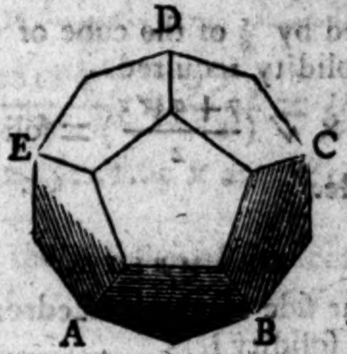
$\frac{\sqrt{3}+\sqrt{15}}{2}$ = diameter of the circumſcribing ſphere, and aD $\frac{\sqrt{3}+\sqrt{15}}{4} = R$ = radius of the circumſcribing ſphere.

Again, the angle FOn contains 72°, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$; and the angle OFn is 54°, whoſe ſine is $\frac{1+\sqrt{5}}{4}$; whence, by trigonometry, $\frac{1}{4}\sqrt{10+2\sqrt{5}} : \frac{1+\sqrt{5}}{4} :: En\,(aD) : OF = aD\,\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} = aD\sqrt{\frac{5+\sqrt{5}}{10}} = aD\sqrt{\frac{1}{2}+\frac{1}{10}\sqrt{5}}$.

But ſince the radius of the circumſcribing ſphere is the hypothenuſe of a right angled triangle, whoſe two legs are OF and the radius of the inſcribed ſphere, we ſhall have $\sqrt{R^2-OF^2} = \sqrt{(\frac{1}{4}\sqrt{3+\sqrt{15}})^2 aD^2 - \overline{\frac{1}{2}+\frac{1}{10}\sqrt{5}}\,aD^2)} = aD\sqrt{\frac{25+11\sqrt{5}}{40}} = \sqrt{\frac{5}{8}+\frac{11}{40}\sqrt{5}}\,r$ = radius of the inſcribed ſphere.

And becauſe the ſolid is compoſed of 12 equal pentagonal pyramids, each of whoſe baſes are, by problem the 8th, $= \frac{5aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$; therefore $\frac{60aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}} \times \frac{1}{3}r = \frac{60aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}} \times \frac{aD}{3}\sqrt{\frac{25+11\sqrt{5}}{40}} = 5aD\sqrt{\frac{47+21\sqrt{5}}{40}}$ $\frac{21\sqrt{5}}{40}$ = ſolidity of the dodecaedron. Q. E. D.

If L be put for the linear ſide, then will $15L^2\sqrt{\frac{5+2\sqrt{5}}{5}}$ = ſurface of the dodecaedron.



$$\sqrt{\frac{21\sqrt{5}+47}{40}} \times 27 \times 5 = \sqrt{\frac{21 \times 2.23606+47}{40}} \times 27 \times 5 = \sqrt{\frac{46.95726+47}{40}} \times 135 = 206.901$$ *ſolidity required.*

2. The linear ſide of a dodecaedron is 1; what is the ſolidity? *Anſ.* 7.6631.

PROBLEM IV.

To find the ſolidity of an icoſaedron.

RULE.*

To 5 times the ſquare root of 5 add 7, and divide the ſum by 2; then the ſquare root of this quotient being

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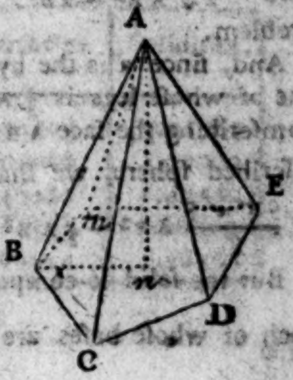


* *Demon.* Let A be a ſolid angle of the icoſaedron, formed by 5 faces, or triangles, whoſe baſes make the pentagon B C D E *m*.

Then, if a perpendicular be demitted from A upon the pentagonal plane B C D E *m*, it will fall into the centre *n*, and B *n*, by the demonſtration of the laſt problem, will be $= AB\sqrt{\frac{5+\sqrt{5}}{10}}$, and the radius of the circle circumſcribing one of the faces A B C $= \frac{1}{3} AB \sqrt{3}$.

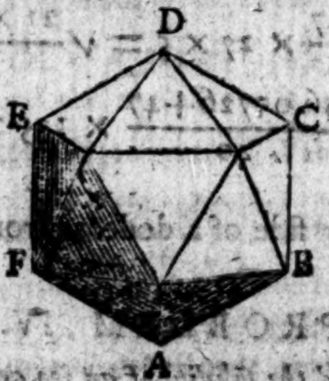
Now

being multiplied by $\frac{5}{6}$ of the cube of the linear ſide will give the ſolidity required.

That is $\frac{5}{6}s^3 \times \sqrt{\left(\frac{7+3\sqrt{5}}{2}\right)}$ = ſolidity when s = to the linear ſide.

EXAMPLES.

1. The linear ſide of the icoſaedron ABCDEF is 3: what is the ſolidity?



$\sqrt{3}$

But the radius of the circumſcribing ſphere is $R = \frac{BA^2}{2An} = \frac{BA^2}{2\sqrt{AB^2 - Bn^2}} = AB\sqrt{\frac{5+\sqrt{5}}{8}}$, found as in the laſt problem.

And, ſince R is the hypothenuſe of a right angled triangle, one of whoſe legs is $\frac{1}{3}AB\sqrt{3}$, the radius of the circle circumſcribing the face ABC, and the other r, the radius of the inſcribed ſphere, we ſhall have $r = \sqrt{R^2 - \frac{1}{3}(AB\sqrt{3})^2} = \sqrt{\frac{5+\sqrt{5}}{8}AB^2 - \frac{1}{3}AB^2} = AB\sqrt{\frac{7+3\sqrt{5}}{24}}$.

But the ſolid is compoſed of 20 equal triangular pyramids, each of whoſe baſes are $= \frac{AB^2}{4}\sqrt{3}$ by problem 8; there-

fore

$$\sqrt{\frac{3\sqrt{5}+7}{2}} \times \frac{5 \times 3^3}{6} = \sqrt{\frac{3 \times 2.23606 + 7}{2}} \times \frac{5 \times 27}{6} = \sqrt{\frac{7.0818+7}{2}} \times \frac{5 \times 9}{2} = \sqrt{\frac{14.0818}{2}} \times \frac{45}{2}$$

$= \sqrt{7.0409} \times 22.5 = 2.65 \times 22.5 = 95.625 =$ *ſolidity required.*

2. Required the ſolidity of an icoſaedron, whoſe linear ſide is 1? *Anſ.* 2.18169499o5.

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fore $\frac{20\,AB^2}{4}\sqrt{3} \times \frac{1}{3}r = 5\,AB^2\sqrt{3} \times \frac{AB}{3}\sqrt{\frac{7+3\sqrt{5}}{24}} = \frac{5}{6}\,AB^3\sqrt{\frac{7+3\sqrt{5}}{2}} =$ ſolidity of the icoſaedron. Q. E. D.

If L be put for the linear ſide, then will $5L^2\sqrt{3} =$ ſurface of the icoſaedron.

Note. The ſuperficies and ſolidity of any of the 5 regular bodies may be found as follows.

Rule. 1. Multiply the tabular area by the ſquare of the linear edge, and the product will be the ſuperficies.

2. Multiply the tabular ſolidity by the cube of the linear edge, and the product will be the ſolidity.

Surfaces and Solidities of the Regular bodies.

N° of Sides.	Names.	Surfaces.	Solidities.
4	Tetraedron	1.73205	0.11785
6	Hexaedron	6.00000	1.00000
8	Octaedron	3.46410	0.47140
12	Dodecaedron	20.64573	7.66312
20	Icoſaedron	8.66025	2.18169

OF

CYLINDRIC RINGS.

PROBLEM I.

To find the convex ſuperficies of a cylindric ring.

RULE.*

TO the thickneſs of the ring add the inner diameter, and this ſum being multiplied by the thickneſs, and the product again by 9.8696 will give the ſuperficies required.

EXAMPLES.

1. The thickneſs of Ac of a cylindric ring is 3 inches, and the inner diameter cd 12 inches: what is the convex ſuperficies?

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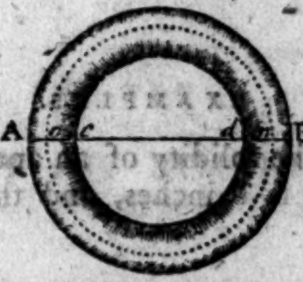
[*] A ſolid ring of this kind is only a bent cylinder, and therefore the rules for obtaining its ſuperficies, or ſolidity, are the ſame as thoſe already given. For let Ac be any ſection of the ſolid perpendicular to its axis on, and then $Ac \times 3.14159$, &c. $=$ circumference of that ſection, and $\overline{Ac + cd}\,(on) \times 3.14159$, &c. $=$ length of the axis on.

Whence $Ac \times \overline{3.14159,\ \&c.} \times \overline{Ac + cd} \times 3.14159,\ \&c. = \overline{Ac+cd} \times Ac \times \overline{3.14159,\ \&c.}^{\,2} = \overline{Ac + cd} \times Ac \times 9.8696$, &c. $=$ ſuperficies required.



$\overline{12+3} \times 3 \times 9.8696 = 15 \times 3 \times 9.8696 = 45 \times 9.8696 = 444.132 =$ *ſuperficies required.*

2. The thickneſs of a cylindric ring is 4 inches, and the inner diameter 18: what is the convex ſuperficies? *Anſ.* 868.52 *ſquare inches.*

3. The thickneſs of a cylindric ring is 2 inches, and the inner diameter 18: what is the convex ſuperficies? *Anſ.* 394.785 *ſquare inches.*

PROBLEM II.

To find the ſolidity of a cylindric ring.

RULE.*

To the thickneſs of the ring add the inner diameter, and this ſum being multiplied by the ſquare of half the thickneſs, and the product again by 9.8696 will give the ſolidity.

---

* *Demon.* $Ac^2 \times .78539$, &c. $= Ac^2 \times \frac{3.14159}{4}$, &c. $= \frac{1}{4} Ac^2 \times 3.14159$, &c. $=$ area of the ſection Ac; and $\overline{Ac+cd}$ $(on) \times 3.14159$, &c. $=$ length of the axis on.

Therefore $\overline{Ac+cd} \times \frac{1}{4} Ac^2 \times \overline{3.14159, \&c.}|^2 = \overline{Ac+ca} \times \frac{1}{4} Ac^2 \times 9.8698$. Q. E. D.

EXAM-

EXAMPLES.

1. What is the ſolidity of an anchor ring, whoſe inner diameter is 8 inches, and thickneſs in metal 3 inches?

$\overline{8+3\times\frac{3}{2}}\,)^2 \times 9.8696 = 11 \times 1.5^2 \times 9.8696 = 11 \times 2.25 \times 9.8696 = 24.75 \times 9.8696 = 244.2726 =$ *ſolidity required.*

2. The inner diameter of a cylindric ring is 18 inches, and its thickneſs 4 inches: what is the ſolidity?
Anſ. 868.5248.

3. Required the ſolidity of a cylindric ring, whoſe thickneſs is 2 inches, and its inner diameter 12.
Anſ. 138.1744.

4. What is the ſolidity of a cylindric ring, whoſe thickneſs is 4 inches, and inner diameter 26?
Anſ. 789.568.

This figure being only a cylinder bent round into a ring, its ſurface and ſolidity may alſo be found as in the cylinder, namely, by multiplying the axis or length of the cylinder by the circumference of the ring, or ſection, for the ſurface, and by the area of a ſection for the ſolidity.

Thus, if $c =$ circumference of the ring, or ſection, $a =$ area of that ſection, and $l =$ length of the axis: then will $cl =$ ſurface of the ring, and $al =$ to its ſolidity. Which rules are the ſame as for the cylinder, and may be eaſily converted into thoſe given in the text.

Theſe rules are indeed ſo obvious, as to render any demonſtration of them altogether unneceſſary.

OF

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimenſions of any ball or ſhell being had from experiments, the weight and dimenſions of any other ball or ſhell, of the ſame kind, may be found by the following rules.

PROBLEM I.

Having the diameter of an iron ſhot given, to find its weight.

RULE.*

Take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that eighth, and the ſum will be the weight required in pounds, *exactly*.

EXAM-

---

* *Demon.* The weight of an iron ſhot whoſe diameter is 4 inches is 9lbs; and as globes are as the cubes of their diameters, therefore $4^3 : 9\text{lbs.} :: D^3$ (diameter being $= D$) : $\frac{9}{64}D^3 =$ weight. But $\frac{9}{64} = \frac{8}{64} + \frac{1}{64} = \frac{1}{8} + \frac{1}{8}$ of $\frac{1}{8}$. Wherefore the weight required is $\frac{1}{8}D^3 + \frac{1}{8}$ of $\frac{1}{8}D^3$.

Q. E. D.

EXAMPLES.

1. The diameter of an iron ſhot is 3.5 inches; what is its weight?

```
        3.5
        3.5
        ---
        175
       105
       ----
      12.25
        3.5
      -----
      6125
     3675
    -------
  8)42.875
    -------
  8)5.359
     .669
    -----
    6.028=lbs. nearly.
```

2. The diameter of an iron ſhot is 6.7 inches: what is its weight? *Anſ.* 42.294*lbs.*

PROBLEM II.

Having the diameter of a leaden ball given to find its weight.

Rule by Logarithms. To 3 times the log. of the diameter, add—1.1480625, and the ſum is the logarithm of the weight required.

RULE.*

Take $\frac{1}{3}$ of the cube of the diameter, and ſrom it ſubtract $\frac{1}{3}$ of this third, and the remainder is the weight required, *nearly.*

EXAMPLES.

1. What is the weight of a leaden ball, whoſe diameter is 3.3 inches?

3.3
3.3

99
99

10.89
3.3

3267
3267

3)35.937

3)11.979
3.993

7.986=8*lbs.* *weight, nearly.*

---

* The weight of a leaden ball, of $4\frac{1}{4}$ inches diameter, is 17lbs. therefore 4.25^3 : 17*lbs.* :: 1 D^3 : $\frac{17}{76.765625} \times D^3 = \frac{2}{9} D^3$ *nearly*, $= (\frac{3}{9} - \frac{1}{9}) \times D^3 = \frac{1}{3} D^3 - \frac{1}{3}$ of $\frac{1}{3} D^3$.

Q. E. D.

Rule by Logarithms. To 3 times the log. of the diameter, add—1.3452821, and the ſum will be the logarithm of the weight in pounds *exactly.*

2. What is the weight of a leaden ball whoſe diameter is 5.24 inches? *Anſ. 32lbs. nearly.*

PROBLEM III.

The weight of an iron ball being given, to find the diameter.

RULE.[*]

Multiply the weight by 7, and to the product add $\frac{1}{9}$ of the weight, and the cube root of the ſum will be the diameter in inches.

EXAMPLES.

1. The weight of an iron ball is 24lbs. what is the diameter?

$$\begin{array}{r} 24 \\ 7 \\ \hline 168 \\ \tfrac{1}{9}\text{ of } 24 = 2.666 \\ \hline 170.666 \end{array}$$

---

[*] *Demon.* By problem I. $\frac{9}{64}D^3 = \text{weight} = w$; therefore $D^3 = \frac{64}{9} \times w = \frac{63}{9}w + \frac{1}{9}w = 7w + \frac{1}{9}w$; and conſequently $D = \overline{7w + \tfrac{1}{9}w}|^{\frac{1}{3}}$. Q. E. D.

The Rule by Logarithms. To the log. of the weight in pounds, add 0.8519375, and $\frac{2}{3}$ of the ſum will be the logarithm of the diameter in inches.

The cube root in any of the following queſtions may alſo be extracted by Logarithms.

† *Let the ſuppoſed root be* 5, *whoſe cube is* 125; *then*

```
125             170.666
  2                   2
---             -------
250             341.332
170.666         125.
-------         -------
420.666    :    466.332    ::    5
                      5
                -------
       420.666)2331.660(5.54 inches = cube root of
               2103330    170.666, or the dia. req.
               -------
                2283300
                2103330
                -------
                 1799700
                 1682664
                 -------
                  117036
```

2. The weight of an iron ball is 12lbs. what is the diameter? *Anſ.* 4.403 *inches.*

PROBLEM IV.

Having the weight of a leaden ball given, to find its diameter.

---

† *The rule here uſed for extracting the cube root is as follows:*

Find a root nearly equal to the true one by trial, and then ſay, as twice the cube of the ſuppoſed reot added to the given number, is to twice the given number added to the cube of the ſuppoſed root, ſo is the ſuppoſed root to the true root, *nearly.*

For a further account of this method, together with its demonſtration, ſee the *Scholar's Guide to Arithmetic,* page 170.

RULE.*

To 4 times the weight, add half the weight, and $\frac{3}{100}$ of half the weight, and the cube root of this ſum will be the diameter, *nearly*.

EXAMPLES.

1. The weight of a leaden ball is 8lbs. what is the diameter?

$$\begin{array}{l} 8\\ 4\\ \hline 32\\ \;\;4=\frac{1}{2}\text{ of }8\\ \hline 36\\ \;\;.12=\frac{3}{100}\text{ of }4\\ \hline 36.12\end{array}$$

Let the ſuppoſed root be 3, *whoſe cube is* 27; *then*

---

* *Demon.* By problem II. $D^3 = \frac{76.765625}{17} \times w = 4w + \frac{1}{2}w + \frac{3}{100}$ of $\frac{1}{2}w$, *very nearly*; therefore $D = \overline{4w + \frac{1}{2}w + \frac{3}{100}\text{ of }\frac{1}{2}w}\Big)^{\frac{1}{3}}$. Q. E. D.

The Rule by Logarithms. To the log. of the weight in pounds, add 0.6547179, and $\frac{1}{3}$ of the ſum will be the logarithm of the diameter, in inches, *exactly*.

$$\begin{array}{rcrcl}
27 & & 36.12 & & \\
2 & & 2 & & \\
\hline
54 & & 72.24 & & \\
36.12 & & 27. & & \\
\hline
90.12 & : & 99.24 & :: & 3 \\
 & & 3 & & \\
\hline
\end{array}$$

$$\begin{array}{l}
90.12)297.72(3.30 = \text{diam. nearly.} \\
27036 \\
\hline
27360 \\
27036 \\
\hline
324
\end{array}$$

2. What is the diameter of a leaden ball whoſe weight is 12lbs? *Anſ.* 3.78 *inches.*

PROBLEM V.

Having the external and internal dimenſions of an iron ſhell given, to find its weight.

RULE.*

Take $\frac{1}{8}$ of the difference of the cubes of the diameter in inches, and $\frac{1}{8}$ of that eighth, and their ſum will be the weight in pounds.

---

* *Demon.* The weight is $\frac{9}{64} \times \overline{D^3 - d^3} = (\frac{8}{64} + \frac{1}{64}) \times \overline{D^3 - d^3} = \frac{1}{8} \times \overline{D^3 - d^3} + \frac{1}{8}$ of $\frac{1}{8} \times \overline{D^3 - d^3}$. Q. E. D.

EXAMPLES.

1. What is the weight of a 13 inch iron bomb-ſhell, the metal being 2 inches thick on a mean?

13
13
39
13
169
13
507
169
2197 = *cube of the external diam.*
9×9×9 = 729 = *cube of the internal diam.*
8)1468
8)183,5
22.9
206,4 = *lbs. weight required.*

2. What is the weight of a 9 inch iron bomb-ſhell, the metal being 1½ inches thick? *Anſ.* 72.14*lbs.*

PROBLEM VI.

To find the number of pounds of powder that a hollow ſhell will hold.

RULE.

RULE.*

Cube the internal diameter in inches, and divide by 59.32, and the quotient is the weight, *nearly*.

EXAMPLES.

1. How many pounds of powder will a hollow ſhell hold, whoſe internal diameter is 9 inches?

$$\begin{array}{r} 9 \\ 9 \\ \hline 81 \\ 9 \\ \hline \end{array}$$

$$\begin{array}{rl} 59.32)729.00(12.28 = & \textit{pounds nearly.} \\ 5932 & \\ \hline 13580 & \\ 11864 & \\ \hline 17160 & \\ 11864 & \\ \hline 52960 & \\ 47456 & \\ \hline 5504 & \end{array}$$

---

* *Demon.* The content of the ſhell is $= D^3 \times .5236$; therefore 31.06 (*in. in 1lb.*) : 1lb. : : $D^3 \times .5236 : \frac{.5236}{31.06} \times D^3 = \frac{D^3}{59.32}$. Q. E. D.

Or the rule may be expreſſed thus. Cube the diameter, and take $\frac{1}{60}$ part of the reſult; alſo $\frac{1}{60}$ part of this $\frac{1}{60}$; and from the

2. How many pounds of powder will a hollow bomb-ſhell hold, whoſe internal diameter is 13 inches?

Anſ. 37*lbs. nearly.*

PROBLEM VII.

To find the dimenſions of a cubical box, to hold a given quantity of powder.

RULE.*

Multiply the weight in pounds by 31.06, and the cube root of the product will give the length of the ſide in inches.

EXAMPLES.

1. What muſt be the length of the ſide of a cubical box that is to hold 15lbs. of powder?

$$\begin{array}{r} 31.06 \\ 16 \\ \hline 15530 \\ 3106 \\ \hline 465.90 \end{array}$$

---

the ſum of theſe two quotients, ſubtract ⅓ of the laſt quotient, and the remainder is the anſwer, *nearly*.

Rule by Logarithms. To 3 times the log. of the diameter in inches, add—2.2267989, and the ſum is the logarithm of the quantity of pounds required.

* *Demon.* As 31.06 inches : 1lb. :: s^3 (the cube of the ſide) : w (the weight); whence $s = \overline{31.06 \times w}|^{\frac{1}{3}}$. Q. E. D.

Rule by Logarithms. To ⅓ of the log. of the weight in pounds, add 0.4974005, and the ſum will be the logarithm of the ſide in inches.

* *Let 7.5 be the ſuppoſed root, whoſe cube is* 421.875; *then*

```
421.875          465.9
      2              2
-------          -----
843.750          931.8
465.90           421.875
--------         -------
1309.650   :     1353.675  ::  7.5
                      7.5
                 --------
                  6768375
                 9475725
                 --------
1309.650)10152.5625(7.75=ſide req.
                 9167550
                 -------
                 9850125
                 9167550
                 -------
                  6825750
                  6548250
                  -------
                   277500
```

2. What is the ſide of a cubical box, that is to hold 12lbs. of powder? *Anſ.* 7.19 *inches.*

PROBLEM VIII.

Given the length, breadth, and depth of a rectangular box, to find how many pounds of powder will fill it.

---

* The cube root may here alſo be found by Logarithms.

RULE.*

Multiply the length, breadth, and depth in inches together; and this product being multiplied again by .0322 will give the anſwer in pounds, *nearly*.

But if greater exactneſs be required, ſubtract $\frac{1}{4000}$ part of the product of the three dimenſions from the laſt reſult, and the remainder will be the anſwer, *very nearly*.

EXAMPLES.

1. The length of a rectangular box is 15 inches, the breadth 13 inches, and the depth 5 inches: how many pounds of powder will it hold?

$$\begin{array}{r}
15 \\
13 \\
\hline
45 \\
15 \\
\hline
195 \\
5 \\
\hline
975 \\
.0322 \\
\hline
1950 \\
1950 \\
2925 \\
\hline
31.3950
\end{array} = \textit{pounds required.}$$

2. The

---

[*] *Demon.* Let L, B, D, = length, breadth, and depth reſpectively.

Then $\frac{LBD}{31.06}$ = weight in pounds = $LBD \times .03219574 = (.0322 - \frac{1}{4000}) \times LBD$. Q. E. D.

2. The length of a rectangular box is 1 foot, the breadth 9 inches, and the depth 4 inches: how many pounds of powder will it hold? *Anſ.* 13.9104*lbs.*

PROBLEM IX.

Given the diameter and length of a hollow cylinder, to find how many pounds of gunpowder will fill it.

RULE.*

Multiply the ſquare of the diameter by $\frac{1}{10}$ of the length; and then take $\frac{1}{4}$ of the product, $\frac{1}{90}$ part of this fourth, and $\frac{1}{90}$ part of this laſt, and the ſum of all theſe parts will be the anſwer in pounds, *nearly*.

Rule by Logarithms. Add the logarithms of the length, breadth, and depth in inches, and the conſtant logarithm —2.5077983 together; and the ſum will be the logarithm of the anſwer in pounds, *exactly*.

* *Demon.* $\frac{D^2 \times .78539 \times L}{31.06} = w = \frac{9}{356} D^2 L$, *nearly*.

But $\frac{9}{356} = \frac{9}{360-4} = \frac{9}{360} + \frac{9}{360} \times \frac{4}{360} + \frac{9}{360} \times \frac{4}{360} \times \frac{4}{360}$, &c. $= \frac{1}{10} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{90} + \frac{1}{4} \times \frac{1}{90} \times \frac{1}{90}$, &c.

Hence $w = D^2 \times \frac{L}{10} \times (\frac{1}{4} + \frac{1}{4} \times \frac{1}{90} + \frac{1}{4} \times \frac{1}{90} \times \frac{1}{90})$, *nearly*. Q. E. D.

The two firſt terms are near enough for common uſe.

Rule by Logarithms. To twice the log. of the diameter, add the logarithm of the length of the cylinder, and the conſtant logarithm—2.4028885, and the ſum will be the logarithm of the weight in pounds, *exactly*.

EXAM-

EXAMPLES.

1. The diameter of a hollow cylinder is 4 inches, and the length 1 foot: how many pounds of powder will it hold?

$$\begin{array}{r l}
4 & \\
4 & \\
\hline
16 & = \textit{ſquare of the diameter.} \\
1.2 & = \tfrac{1}{10} \textit{ of the length.} \\
\hline
32 & \\
16 & \\
\hline
4)\,19.2 & \\
\hline
4.8 & \\
.0533 & = \tfrac{1}{90} \textit{ of } 4.8 \\
.0005 & = \tfrac{1}{90} \textit{ of } .0533 \\
\hline
4.8538 & \textit{pounds the anſwer.}
\end{array}$$

2. The diameter of a hollow cylinder is 6 inches, and its length 10 inches: how many pounds of powder will it hold? *Anſ.* 9.1 *lbs.*

PROBLEM X.

Given the diameter of a hollow cylinder, to find what length of it will be filled by a given quantity of powder.

RULE.*

1. Divide the weight in pounds by the ſquare of the diameter in inches. 2. Mul-

---

* *Demon.* By the laſt problem $L = \frac{356}{9} \times \frac{w}{D^2} = 39\frac{5}{9} \times \frac{w}{D^2} = (40 - \frac{4}{9}) \times \frac{w}{D^2} = (40 - \frac{3}{9} - \frac{1}{9}) \times \frac{w}{D^2} = \frac{40w}{D^2} - (\frac{1}{3} \times \frac{w}{D^2} - \frac{1}{3} \text{ of } \frac{1}{3} \times \frac{w}{D^2})$ *nearly.* Q. E. D.

Rule

2. Multiply the quotient by 40, and from the product ſubtract $\frac{1}{3}$ of the quotient.

3. From the remainder take $\frac{1}{3}$ of that third, and this laſt remainder will be the anſwer, *nearly*.

EXAMPLES.

1. The diameter of a hollow cylinder is 1 foot: what length of it will be filled by 10 pounds of powder?

12)10
12).8333
.0694416=10*lbs. divided by the ſquare of* 12.
40
2.7776640
.0231472=$\frac{1}{3}$ *of* .0694416
2.7545168
.0077157$\frac{1}{3}$= *of* .0231472
2.7468011=*height in inches required.*

2. The diameter of a hollow cylinder is 6 inches: what length of it will be filled by 12lbs. of powder?

Anſ. 13.185.

Rule by Logarithms. To the conſtant logarithm 1.597111, add the logarithm of the given quantity of powder in pounds, and alſo the arithmetical complement of twice the logarithm of the diameter in inches; and the ſum will be the logarithm of the length in pounds.

Theſe rules, and their inveſtigations, were given me by Mr. *Reuben Burrow*.

OF THE PILING OF BALLS AND SHELLS.

IRON *ſhot and ſhells* are uſually piled in horizontal courſes, in a pyramidical or wedge like form, the baſe being either an equilateral triangle, a ſquare or a rectangle. In the triangle and ſquare, the pile finiſhes in a ſingle ball; but in the rectangle, the top is a ſingle row of balls.

In triangular and ſquare piles the number of horizontal rows or courſes is always equal to the number of ſhot on one ſide, in the bottom row. But, in rectangular piles, the number of courſes is equal to the number of ſhot in the breadth of the bottom row; and the number in the top row, leſs one, is the difference between the number in the length and breadth of the bottom row.

PROBLEM I.

To find the number of ſhot in a finiſhed triangular pile.

RULE.*

Multiply the number in one ſide of the bottom row plus two, by that number plus one; and this product being again multiplied by $\frac{1}{6}$ of the number in the ſaid row, will give the anſwer required.

---

* *Demon.* In triangular piles, each horizontal courſe is a triangular number, produced by taking the ſucceſſive ſums of the

EXAMPLES.

1. Required the number of ſhot in a finiſhed triangular pile, the number in one ſide of the baſe being 20?

22 = *number plus* 2
21 = *number plus* 1
—
22
44
—
462
20
—
6)9240
—
1540 = *number of ſhot in the whole pile.*

2. Required the number of ſhot in a finiſhed triangular pile, the number in one ſide of the baſe being 40? *Anſ.* 11480.

---

the numbers 1; 1, 2; 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5, &c. and the whole number of ſhot in ſuch a pile is equal to the ſum of all thoſe triangular numbers, taken as far, or to as many terms, as are equal to the number in one ſide of the bottom courſe.

Thus, if 1, 2, 3, 4, 5, 6, 7, &c. be the natural numbers, then will 1, 3, 6, 10, 15, 21, 28, be the triangular numbers; or the number of ſhot in each courſe from the top.

But the ſum of the ſeries $1+3+6+10+15+21+28$, &c. to n terms is $= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$ (by *Simpſon*'s *Algebra*, page 216) $= \overline{n+2} \times \overline{n+1} \times \frac{n}{6}$; which is the ſame as the rule. Q. E. D.

PROBLEM II.

To find the number of ſhot in a finiſhed ſquare pile.

RULE.*

Multiply the number in one ſide of the bottom row plus one, by double that number plus one; and this product being again multiplied by $\frac{1}{6}$ of the ſaid number, will give the anſwer required.

EXAMPLES.

1. Required the number of ſhot in a finiſhed ſquare pile, the number in one ſide of its baſe being 20.

```
  41=double of 20, plus 1.
  21=20 plus 1.
  --
  41
 82
 ---
 861
  20
 -------
6)17220
 -------
 2870=number of ſhot in the pile.
```

2. Required

---

* *Demon.* In ſquare piles, each horizontal courſe is a ſquare number, produced by taking the ſquare of the number in its ſide; and the whole number of ſhot in ſuch a pile is equal to the ſum of all thoſe ſquares, beginning at one, and proceeding as far as the number in the ſide of the bottom courſe.

Thus, if 1, 2, 3, 4, 5, 6, 7, &c. be the natural numbers, then will 1, 4, 9, 16, 25, 36, 49, &c. be the ſquare numbers, or the number of ſhot in each courſe from the top.

But

2. Required the number of ſhot in a finiſhed ſquare pile, one ſide of the lower tier having 40 ſhot in it.

Anſ. 22140.

PROBLEM III.

To find the number of ſhot in a finiſhed rectangular pile.

RULE.*

1. To twice the length of the uppermoſt row add the number of courſes leſs one, and multiply the ſum by the number of courſes plus 1.

But the ſum of the ſeries $1+4+9+16+25+36+49$, &c. to n terms is $= \frac{n \times (n+1) \times (2n+2)}{6}$ (by *Simpſon*'s *Algebra*, page 206) $= \overline{n+1} \times \overline{2n+2} \times \frac{n}{6}$, which is the ſame as the rule. Q. E. D.

* *The inveſtigation of this rule* is given by *Mr. Simpſon*, in page 209 of his Algebra thus: Let $m+1$ and $p+1$ repreſent the length and breadth of the uppermoſt rank, and n the number of ranks one above another.

Then will $\overline{m+1} \times \overline{p+1} + \overline{m+2} \times \overline{p+2} + \overline{m+3} \times \overline{p+3}$, &c. be the number of ſhot in each of the rectangular courſes of the pile.

And $\overline{m+1} \times \overline{p+1} + \overline{m+2} \times \overline{p+2} + \overline{m+3} \times \overline{p+3} \ldots \ldots + \overline{m+n} \times \overline{p+n} =$ number of ſhot in the pile, whether it be whole, or broken.

But $\overline{m+1} \times \overline{p+1} + \overline{m+2} \times \overline{p+2} + \overline{m+3} \times \overline{p+3} \ldots\ldots + \overline{m+n} \times \overline{p+n} = n \times (mp + \frac{n+1}{2} \times \overline{m+p} + \frac{(n+1) \times (2n+1)}{6}$ $= \frac{n}{4} \times (\overline{2m+n+1} \times \overline{2p+n+1} + \frac{(n+1) \times (n-1)}{3}$ which is the ſame as the rule. Q. E. I.

2. Multiply the number of courſes leſs one by that number more one, and add $\frac{1}{3}$ of this product to the former.

3. Take $\frac{1}{4}$ of this laſt ſum and multiply it by the number of courſes, and it will give the anſwer required.

EXAMPLES.

1. How many ſhot are there in a finiſhed rectangular pile of 15 courſes, the number in the uppermoſt row being 32?

32
2
64 = *twice the uppermoſt row.*
14 = *number of courſes leſs* 1.
78
16
468
78
1248
14 = *number of courſes leſs* 1.
16 = *number of courſes more* 1.
84
14
3)224
$74\frac{2}{3}$
1248
$1322\frac{2}{3}$

4)$1322\frac{2}{3}$

$$
\begin{array}{r}
4)\,1322\tfrac{2}{3} \\
\hline
330\tfrac{2}{3} \\
15 \\
\hline
1660 \\
330 \\
\hline
4960
\end{array}
$$

4960 = *number of ſhot in the pile.*

2. How many ſhot are there in a finiſhed rectangular pile of 20 courſes, the number in the top row being 40? *Anſ.* 11060.

The number of ſhot in any broken pile may be obtained, by computing what the whole finiſhed pile would contain, and alſo what the pile taken away contained; in which caſe the remainder muſt be the number in the unfiniſhed part as required.

The rules for finding the number of ſhot in the different kinds of piles, may be expreſſed algebraically thus:

$$\frac{n}{6} \times \overline{n+1} \times \overline{n+2} = \text{N}^{\circ} \text{ in the triangular pile.}$$

$$\frac{n}{6} \times \overline{n+1} \times \overline{2n+1} = \text{N}^{\circ} \text{ in the ſquare pile.}$$

$$\frac{n}{6} \times (n+1) \times (3m-n+1) = \text{N}^{\circ} \text{ in the rectangular pile.}$$

Where n denotes the number in one ſide of the bottom row, for the triangular and ſquare piles. And in the rectangular pile it denotes the breadth of the bottom row, and m its length. Which rule, for the latter caſe, is ſomething more convenient than that given in the text.

OF

ARTIFICERS WORK.

ARTIFICERS eſtimate, or compute, the value of their works by different meaſures, *viz**.

1. *Glazing*, and *Maſon*'s flat work, &c. by the foot.

2. *Painting*, *Plaſtering*, *Paving*, &c. by the yard.

3. *Flooring*, *Partitioning*, *Roofing*, *Tiling*, &c. by the ſquare of 100 feet.

4. *Brickwork*, &c. by the rod of 16½ feet, whoſe ſquare is 272¼.

The meaſures made uſe of in theſe works are contained in the following table:

12 inches	make	1 lineal foot
144 ſquare inches		1 ſquare foot
9 ſquare feet		1 ſquare yard
100 ſquare feet		a ſquare
272¼ ſquare feet, or 30¼ ſquare yards		1 rod, perch, or ſquare pole.

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* The beſt method of taking the dimenſions of all ſorts of artificers work is by feet, tenths and hundredths; becauſe the computations may then be performed by common multiplication, or by the ſliding rule, hereafter deſcribed.

OF

BRICKLAYERS WORK*.

BRICKLAYERS compute or value their work at the rate of a brick and a half thick, and, if a wall be more or leſs than this ſtandard, it muſt be reduced to it, as follows:

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* *Note.* In practice it is uſual to divide the ſquare feet by 272 only, omitting the ¼.

The uſual way to take the dimenſions of a building, is to meaſure half round its middle, on the outſide, and half round it on the inſide; and this will give the true compaſs, in which the thickneſs of the wall is included.

When the height of the building is unequal, take ſeveral different altitudes, and their ſum, being divided by the number you have taken, may be conſidered as the mean height.

To meaſure a chimney ſtanding by itſelf, without any party-wall adjoining; girt it about for the length, and reckon the height of the ſtory for the breadth; but if it ſtands againſt a wall, you muſt meaſure it round to the wall for the girt, and take the height as before.

When the chimney is wrought upright from the mantle-tree to the cieling, the thickneſs muſt always be the ſame with the jaumbs; and nothing is ever deducted for the vacancy between the floor and the mantle-tree, becauſe of the gathering of the breaſt and wings to make room for the hearth in the next ſtory.

To meaſure chimney ſhafts, or that part which appears above the roof; girt them with a line, about the leaſt place for the length, and take the height for the breadth; and if they be four inches thick, ſet down the thickneſs at one brick-work; but if they be nine inches thick, reckon it at a brick and a half, in conſideration of the plaſtering and ſcaffolding.

All

RULE.

Multiply the ſuperficial content of the wall in feet, by the number of half bricks in the thickneſs, and $\frac{1}{3}$ of that product will be the content required.

EXAMPLES.

1. How many ſquare rods are there in a wall $52\frac{1}{2}$ feet long, 12 feet 9 inches high, and $2\frac{1}{2}$ bricks thick?

By Decimals.

```
         52.5 length
        12.75=height
      -------
         2625
        3675
       1050
       525
      -------
  272)669.375(2.4609
      544          5 half bricks
      -----    ---------
      1253  3)12.3045
      1088    ---------  ro. fe. in.
    -------   4.1015=4  27   7 the anſwer.
       1657
       1632
      -------
         2550
         2448
        -----
          102
```

By

All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the value of

By Croſs Multiplication.

feet in.
52 6
12 9

630 0
39 4 6

272)669 4 6

2 124 4
5

3)12 82 8

4 27 7 *as before.*

2. How many ſquare rods are there in a wall 62½ feet long, 14 feet 8 inches high, and 2½ bricks thick?

ro. fe. in. p.
Anſ. 5 167 9 4

3. If each ſide wall of a building be 45 feet long on the outſide, each end wall 15 feet broad on the inſide, the height of the building 20 feet, and the gable at each end of the wall 6 feet high, the whole being 2 bricks thick; what is the true content in ſtandard rods? *Anſ.* 12.1761.

---

of their workmanſhip is added to the bill at the ſtated rate agreed on.

There are alſo other allowances to be made to the workman, ſuch as thoſe for returns of angles made by two adjoining walls, and double meaſure for feathered gable ends, &c.

All ornamental work is generally valued by the foot ſquare, ſuch as arches, doors, architraves, frizes, cornices, &c. But carved mouldings, &c. are often agreed for by the running foot, or lineal meaſure.

OF

MASONS WORK.

TO Maſonry belongs all ſorts of ſtone work, and the meaſure made uſe of is a foot, either ſuperficial or ſolid.

Walls, blocks of marble or ſtone, columns, &c. are meaſured by the ſolid foot; and pavements, ſlabs, chimney-pieces, &c. by the ſuperficial foot.*

EXAMPLES.

1. Required the ſolid content of a wall whoſe length is 48 feet 6 inches, its height 10 feet 9 inches, and thickneſs 2 feet.

By Decimals.

```
        48.5
        10.75
      ------
        2425
       3395
      4850
      ------
      521.375
            2
      ------
     1042.750 the anſwer.
```

---

* Solid meaſure is principally uſed for materials, and the ſuperficial for workmanſhip.—In the ſolid meaſure the true length, breadth, and thickneſs, are taken, and multiplied continually together. And in the ſuperficial meaſure, the length and breadth of every part of the projection muſt be taken, as it appears without the general upright face of the building. All

By Croſs Multiplication.

feet	*in.*	
48	6	
10	9	
485	0	
36	4	6
521	4	6
2	0	0
1042	9	0 *the ſame as before.*

2. Required the ſolid content of a wall whoſe length is 53 feet 6 inches, its height 12 feet 3 inches, and its thickneſs 2 feet. *Anſ.* 1310 *feet* 9 *in.*

3. What is a marble ſlab worth, whoſe length is 5 feet 7 inches, and breadth 1 foot 10 inches, at 6s. per foot? *Anſ.* 3*l.* 1*s.* 5*d.*

4. What is the ſolid content of a wall, whoſe length is 60 feet 9 inches, its height 10 feet 3 inches, and its thickneſs 2½ feet? *Anſ.* 1556.71875 *feet.*

5. In a chimney-piece, ſuppoſe the

	fe.	*in.*
Length of the mantle and ſlab each,	4	6
Breadth of both together,	3	2

	fe.	*in.*
Length of each jamb,	4	4
Breadth of both together,	1	9

What will be the content of the chimney-piece? *Anſ.* 21 *feet* 10 *in.*

All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed; but this deduction is only to be made with regard to materials, for the value of the workmanſhip is to be added to the workman's bill at the ſtated rate only.

OF

CARPENTERS AND JOINERS WORK*.

CARPENTERS and Joiners work is that of flooring, partitioning, roofing, &c. and is meaſured by the ſquare of 100 feet.

EXAM-

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* *Note.* Large and plain articles are uſually meaſured by the foot, or yard, &c. ſquare; but inriched mouldings, and ſome other articles, are often eſtimated by running or lineal meaſure, and ſome things are rated by the piece.

In meaſuring of joiſts it is to be obſerved, that only one of their dimenſions is the ſame with that of the floor, and the other will exceed the length of the room by the thickneſs of the wall, and $\frac{1}{3}$ of the ſame, becauſe each end is let into the wall about $\frac{2}{3}$ of its thickneſs.

No deductions are made for hearths, on account of the additional trouble and waſte of materials.

Partitions are meaſured from wall to wall for one dimenſion, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In meaſuring of Joiners work, the ſtring is made to ply cloſe to every part of the work over which it paſſes.

The meaſuring of centering for cellars is found by making a ſtring paſs over the ſurface of the arch for the breadth, and taking the length of the cellar for the length; but in groin-centering, it is uſual to allow double meaſure, on account of their extraordinary trouble.

In roofing, the length of the houſe in the inſide, together with $\frac{2}{3}$ of the thickneſs of one gable, is to be conſidered as the length, and the breadth is equal to double the length of a ſtring which is ſtretched from the ridge down to the rafter, along the eaves board, till it meets with the top of the wall.

For

EXAMPLES.

1. If a floor be 57 feet 3 inches long, and 28 feet 6 inches broad: how many ſquares will it contain?

By Decimals.

57.25
28.5

28625
45800
11450

16|31.625

By

For ſtair-caſes, take the breadth of all the ſteps, and make a line ply cloſe over them, from the top to the bottom, and multiply the length of this line by the length of a ſtep for the whole area.—By the length of a ſtep is meant the length of the front and the returns at the two ends, and by the breadth is to be underſtood the girt of its two upper ſurfaces, or the tread and riſer.

For the baluſtrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-poſt, for the length: and twice the length of the baluſter upon the landing, with the girt of the hand-rail, for the breadth.

For wainſcotting, take the compaſs of the room for the length; and the height from the floor to the cieling, making the ſtring ply cloſe into all the mouldings, from the breadth.—Out of this muſt be made deductions for windows, doors, and chimneys, &c. but workmanſhip is counted for the whole, on account of the extraordinary trouble.

For doors, it is uſual to allow for their thickneſs, by adding it into both the dimenſions of length and breadth, and then multiplying them together for the area.—If the door be pannelled on both ſides, take double its meaſure for the work-

By Croſs Multiplication.

feet	*in.*	
57	3	
28	6	
1603	0	
28	7	6
16|31	7	6

2. A floor is 53 feet 6 inches long, and 47 feet 9 inches broad: how many ſquares will it contain? *Anſ.* 25 *ſq. and* 54 *feet.*

3. A partition is 91 feet 9 inches long, and 11 feet 3 inches broad: how many ſquares will it contain? *Anſ.* 10 *ſq. and* 32 *feet.*

4. If a houſe within the walls be 44 feet 6 inches long, and 18 feet 3 inches broad: how many ſquares of roofing will cover it? *Anſ.* 12 *ſq. and* 18 *feet.*

5. If a houſe meaſure within the walls 52 feet 8 inches in length, and 30 feet 6 inches in breadth, and the roof be of a true pitch, what will it coſt roofing at 10*s.* 6*d.* per ſquare? *Anſ.* 12*l.* 12*s.* 11¾*d.*

---

manſhip; but if one ſide only be pannelled, take the area and its half for the workmanſhip.

For the ſurrounding architrave, gird it about the outermoſt part for its length; and meaſure over it, as far as it can be ſeen when the door is open, for the breadth.

Window-ſhutters, baſes, &c. are meaſured in the ſame manner.

In the meaſuring of roofing for workmanſhip alone, all holes for chimney ſhafts and ſky-lights are generally deducted.

But in meaſuring for work and materials, they commonly meaſure in all ſky-lights, luthern-lights, and holes for the chimney ſhafts, on account of their trouble and waſte of materials.

OF

SLATERS AND TILERS WORK.

IN theſe works, the content of a roof is found by multiplying the length of the ridge by the girt from eave to eave; and, in ſlating, allowance muſt be made for the double row at the bottom.

In taking the girt, the line is made to ply over the loweſt row of ſlates, and returned up the under-ſide till it meet with the wall or eaves-board; but in tiling, the line is ſtretched down only to the loweſt part, without returning it up again.

Double meaſure is generally allowed for hips, vallies, gutters, &c. but no deductions are made for chimnies.

EXAMPLES.

1. The length of a ſlated roof is 45 feet 9 inches, and its girt 34 feet 3 inches: what is the content?

By Decimals.

```
     45.75
     34.25
    ------
     22875
     9150
   18300
  13725
  ---------
9)1566.9375
  ---------
   174.104
```

By Croſs Multiplication.

	feet	*in.*		
	45	9		
	34	3		
	1555	6		
	11	5	3	
9)	1566	11	3	
	174	0	11	3

Anſ. 174 *yards.*

2. What will the tiling a barn coſt at 25*s.* 6*d.* per ſquare, the length being 43 feet 10 inches, and the breadth 27 feet 5 inches, on the flat, the eave-boards projecting 16 inches on each ſide? *Anſ.* 24*l.* 9*s.* 5½*d.*

In angles formed in a roof, running from the ridge to the eaves, that angle of the roof, which bends inwards, is called a *valley*; and the angle bending outwards is called a *hip*. And in tiling and ſlating, it is common to add the length of the vallies to the content in feet; and ſometimes alſo the hips are added.

In ſlating it is common to reckon the breadth of the roof 2 or 3 inches broader than what it meaſures, becauſe the firſt row is almoſt covered by the ſecond; and this is done ſometimes when a roof is tiled.

Note. Sky-lights and chimney-ſhafts are always deducted; but they ſeldom deduct luthern lights, or garret windows on the roof; for the covering them is reckoned equal to the hole in the roof.

In all works of this kind the content is computed, either in yards of 9 ſquare feet, or in ſquares of a hundred feet, and the ſame allowance of hips and vallies is to be made as in roofing.

It is cuſtomary to reckon the flat and half of any building within the walls, for the meaſure of the roof of that building, when the roof is of a *true pitch*, or ſo that the rafters are ¾ of the breadth of the building.

PLASTERERS

PLASTERERS WORK.

PLASTERERS work is of two kinds, viz. plaſtering upon laths, called cieling; and plaſtering upon walls, called rendering; and theſe different kinds muſt be meaſured ſeparately, and their contents collected into one ſum.

Note. Proper deductions muſt be made for doors, windows, &c.—And in meaſuring between quarters, there is commonly $\frac{1}{5}$ part of the whole area allowed; but when rendering between quarters is whitened or coloured, there is $\frac{1}{5}$ part to be added to the whole for the ſides of the quarters and braces.

EXAMPLES.

If a cieling be 59 feet 9 inches long, and 24 feet 6 inches broad: how many yards does it contain?

By Decimals.

```
     59.75
      24.5
   -------
     29857
    23900
   119500
   -------
9)1463.875
   -------
   162.652
```

Plaſterers plain work is meaſured by the ſquare foot, or yard of 9 ſquare feet, and enriched mouldings, &c. by running or lineal meaſure.

By Croſs Multiplication.

feet *in.*
59 9
24 6

1434 0
29 10 6

9)1453 10 6

162 5 10 6

Anſ. 162 *yards* 5 *feet.*

2. If the partitions between rooms be 141 feet 6 inches about, and 11 feet 3 inches high; how many yards do they contain? *Anſ.* 176.87.

3. There is a quantity of partitioning that meaſures 234 feet 8 inches about, and 14 feet 6 inches high, and is rendered between quarters; the lathing and plaſtering of which will be 8*d.* per yard, and the whiting 2*d.* per yard: what will the whole come to? *Anſ.* 13*l.* 17*s.* 2$\frac{3}{4}$*d.*

4. The length of a room is 14 feet 5 inches, its breadth 13 feet 2 inches, and height 9 feet 3 inches to the under ſide of the cornice, whoſe girt is 8$\frac{1}{2}$ inches, and its projection 5 inches from the wall on the upper part next the cieling: what will be the quantity of plaſtering, ſuppoſing there are no deductions but for one door, whoſe ſize is 7 feet by 4 feet? *Anſ.* 53 *yards* 5 *feet* 3 *inches, of rendering,* 18 *yards* 5 *feet* 6 *inches, of cieling, and* 39 *feet* 0$\frac{1}{11}$ *inches, of cornice.*

The work is chiefly of two kinds; viz. plaſtering upon laths, called cieling; and plaſtering upon walls, called rendering; all the different kinds are to be meaſured ſeparately; and then the contents of the ſame kind, or the ſame number of coats muſt be collected together into one ſum for the content of the whole.

OF

PAINTERS WORK.

PAINTERS work is meaſured in the ſame manner as that of Joiners; and in taking the dimenſions, the line muſt be forced into all the mouldings and corners.

Windows are done at ſo much a-piece, and in carved mouldings, &c. it is cuſtomary to allow double the uſual meaſure.

EXAMPLES.

1. If a room be painted, whoſe height is 16 feet 6 inches, and its compaſs 97 feet 9 inches: how many yards does it contain?

By Decimals.

```
  97.75
   16.5
 -------
  48875
 58650
 9775
 -------
9)1612.875
 -------
 179.208
```

Baluſtrades and moſt other works of that kind are meaſured as in Joiners work, excepting for doors, window-ſhutters, &c. where the Joiner is only allowed the area and half area, but the Painter has always double the area of one ſide, becauſe every part that is painted muſt be meaſured.

By Croſs Multiplication.

feet in.
97 9
16 6

1564 0
48 10 6

9)1612 10 6

179 1 10 6

Anſ. 179 *yards* 1 *foot.*

2. The height of a room is 14 feet 10 inches, and the circumference 21 feet 8 inches, how many ſquare yards does it contain? *Anſ.* 35.

3. Suppoſe a room, that was to be painted at 8*d.* *per* yard, meaſures as follows: The height is 11 *fe.* 7 *in.* the girt or compaſs 74 *fe.* 10 *in.* the door 7 *fe.* 6 *in.* by 3 *fe.* 9 *inches*; five window ſhutters, each 6 *fe.* 8 *in.* by 3 *fe.* 4 *in.* the breaks in the windows 14 *in.* deep, and 8 *fe.* high; the chimney 6 *fe.* 9 *in.* by 5 *fe.* a cloſet, the height of the room, 3½ *fe.* deep, and 4¼ *fe.* in front, with ſhelving, at 22 *fe.* 6 *in.* by 10 *in.* the ſhutters, doors and ſhelves, being all coloured on both ſides: what will the whole come to?

Anſ. 4*l.* 18*s.* 9*d.*

Note. Painters take their dimenſions with a ſtring, and meaſure from the top of the cornice to the floor, girting the ſtring over all the mouldings and ſwellings; and their price is generally proportioned to the number of times they lay on their colour.

All work of this kind is done by the ſquare yard, and every part where the colour lies muſt be meaſured, and eſtimated in the general account of the work.

Deductions are to be made for chimnies, caſements, &c. and the price is generally proportioned to the number of times they lay on their colour.

OF

GLAZIERS WORK.

GLAZIERS take their dimenſions in feet, inches, and parts, and eſtimate their work by the ſquare foot.

In taking the length and breadth of a window, the croſs-bars between the panes are always included.

Windows of every form are meaſured as if they were ſquares, and the greateſt lengths and breadths are conſtantly to be taken, on account of the waſte attending the cutting.

EXAMPLES.

1. If a pane of glaſs be 2 feet 8 inches and 3 quarters long, and 1 foot 4 inches 1 quarter broad: how many feet does it contain?

By Decimals.

The decimal of 8¾ *inches is* .729
And that of 4¼ *inches is* .354

```
     2.729
     1.354
    ------
     10916
    13645
    8187
   2729
  --------
  3.695066
        12
  --------
  8.340792
```

Glaziers generally meaſure their work to a quarter of an inch, and all circular, triangular, &c. windows, are meaſured as if they were ſquares.

By Croſs Multiplication.

feet	*in.*	*pa.*		
2	8	9		
1	4	3		
2	8	9		
	10	11	0	
		8	2	3
3	8	4	2	3

Anſ. 3 *feet* 8 *in.* 4 *pa.*

2. What will the glazing a triangular ſky-light come to at 10*d.* per foot, ſuppoſing the baſe to be 12 feet 6 inches, and the perpendicular height 16 feet 9 inches? *Anſ.* 4*l.* 7*s.* 2¼*d.*

3. There is a houſe with 3 tier of windows, three in a tier; the height of the firſt tier is 7 feet 10 inches; of the ſecond 6 feet 8 inches; of the third 5 feet 4 inches, and the breadth of each 3 feet 11 inches: what will the glazing come to at 14*d.* per foot? *Anſ.* 13*l.* 11*s.* 10½*d.*

4. There is a houſe with 3 tier of windows, three in a tier; the height of thoſe in the lower tier being 7 *fe.* 9 *in.* of thoſe in the middle 6 *fe.* 6 *in.* and of thoſe in the upper tier 5 *fe.* 3¼ *in.* and having alſo a circular window above the door, the greateſt height of which is 1 foot 10½ *in.* and the common breadth of all the windows 3 *fe.* 9 *in.* what will the glazing come to at 13½*d.* per foot? *Anſ.* 12*l.* 14*s.* 11½*d.*

Note. Windows are ſometimes meaſured by taking the dimenſions of one pane, and multiplying it continually by the number of panes; and no allowance is ever made for round or oval windows, as the trouble of cutting them to thoſe ſhapes is more than the value of the glaſs omitted.

OF

PAVIOURS WORK*.

PAVIOURS work is done by the ſquare yard, and the content is ſound by multiplying the length by the breadth.

Or if the dimenſions be taken in feet, and the area be found in the ſame meaſure, the reſult being divided by 9, will give the number of ſquare yards required.

EXAMPLES.

1. What will the paving a rectangular court-yard come to at 3*ſ*. 2*d*. per yard, ſuppoſing the length to be 27 feet 10 inches, and the breadth 14 feet 9 inches?

By Croſs Multiplication.

feet	*in.*		
27	10		
14	9		
389	8		
20	10	6	
9)410	6	6	
45	5	6	6

---

* Plumbers work is generally done by the pound, or hundred weight, and the price is regulated according to the value of the lead at the time the contract is made, or when the work is performed.

				45	5	6	6 *at* 3*s.* 2*d.*
s.	*d.*						
2	0	$\frac{1}{10}$		45			
1	0	$\frac{1}{2}$		4	10		
	2	$\frac{1}{6}$		2	5		
f.	*in.*	*p.*					
3	0	0	$\frac{1}{3}$		7	6	
1	0	0	$\frac{1}{3}$		1	0½	
1	0	0	$\frac{1}{3}$		0	4	
0	6	0	$\frac{1}{2}$		0	4	
0	0	6	$\frac{1}{12}$		0	2	
					0	0	
				7	4	4½ *the anſwer.*	

2. A rectangular court-yard is 42 feet 9 inches long, and 68 feet 6 inches in depth, and a foot-way goes quite through it, of 5 feet 6 inches in breadth; the foot-way is laid with ſtone at 3*s.* 6*d.* per yard, and the reſt with pebbles at 3*s.* per yard: what will the whole come to? *Anſ.* 49*l.* 17*s.* 0¼*d.*

Sheet-lead, uſed in roofing, guttering, &c. is generally between 7 and 12lbs. weight to the ſquare foot.

The following table will ſhew the weight of a ſquare foot to each of theſe thickneſſes.

Thick.	*lbs. ſq. foot.*	*Thick.*	*lbs. ſq. foot.*	*Thick.*	*lbs. ſq. foot.*
$\frac{1}{8}$	7.373	.15	8.848	.18	10.618
.13	7.668	.16	9.438	.19	11.207
.14	8.258	$\frac{1}{6}$	9.831	$\frac{1}{5}$	11.797
$\frac{1}{7}$	8.427	.17	10.028	.21	12.387

OF

VAULTED AND ARCHED ROOFS.

ARCHED *roofs* are either *vaults*, *domes*, *ſaloons*, or *groins*.

Vaulted roofs are formed by arches ſpringing from the oppoſite walls, and meeting in a line at the top.

Domes are made by arches ſpringing from a circular or polygonal baſe, and meeting in a point at the top.

Saloons are formed by arches connecting the ſide walls to a flat roof, or cieling, in the middle.

Groins are formed by the interſection of vaults with each other.

Domes and Saloons rarely occur in the practice of meaſuring, but vaults and groins cover the cellars of moſt houſes.

Vaulted roofs are generally one of the three following ſorts:

1. *Circular roofs*, or thoſe whoſe arch is ſome part of the circumference of a circle.

2. *Elliptical roofs*, or thoſe whoſe arch is ſome part of the circumference of an ellipſis.

3. *Gothic roofs*, or thoſe which are formed by two circular arcs that meet in a point directly over the middle of the breadth, or ſpan of the arch.

PROBLEM I.

To find the ſolid content of circular, elliptic, or gothic vaulted roofs.

RULE.

RULE.*

Multiply the area of one end by the length of the roof, and the product will be the ſolidity required.

EXAMPLES.

1. What is the ſolid content of a ſemi-circular vault, whoſe ſpan is 40 feet, and its length 120 feet?

```
       .7854
        1600=square of 40.
     ───────
     4712400
     7854
     ───────
  2)1256.6400
     ───────
      628.32=area of the end.
        120=length.
     ───────
   75398.40=ſolidity required.
```

2. Required the ſolidity of an elliptic vault, whoſe ſpan is 40 feet, height 12 feet, and length 80?

Anſ. 30159.36 *feet.*

3. What is the ſolid content of a gothic vault, whoſe ſpan is 48, the chord of its arch 48, the diſtance of the arch from the middle of the chord 18, and the length of the vault 18? *Anſ.* 136228.044.

---

* *To find the ſolidity of the materials in either of the arches.*

Rule. From the ſolid content of the whole arch take the ſolid content of the void ſpace, and the remainder will be the ſolidity of the arch.

PROBLEM II.

To find the concave, or convex ſurface, of circular, elliptic, or gothic vaulted roofs.

RULE.*

Multiply the length of the arch by the length of the vault, and the product will be the ſuperficies required.

EXAMPLES.

1. What is the concave ſurface of a ſemi-circular vault, whoſe ſpan is 40 feet, and its length 120?

$$\begin{array}{r} 3.1416 \\ 40 \\ \hline 2)\,125.6640 \\ \hline 62.832 \end{array}$$

62.832 = *length of the arch.*

$$\begin{array}{r} 62.832 \\ 120 \\ \hline 7539.840 \end{array}$$

7539.840 = *concave ſurface required.*

PROBLEM III.

To find the ſolid content of a dome; its height, and the dimenſions of its baſe being known.

---

[*] The convex ſurface of a vault may be found by ſtretching a ſtring over it; but for the concave ſurface this method is not applicable, and therefore its length muſt be found from proper dimenſions.

RULE.

RULE.*

Multiply the area of the baſe by $\frac{2}{3}$ of the height, and the product will be the ſolidity.

EXAMPLES.

1. What is the ſolid content of a ſpherical dome, the diameter of whoſe circular baſe is 60 feet?

$$\begin{array}{r}
.7854 \\
3600 = \textit{square of } 60. \\
\hline
4712400 \\
23562 \\
\hline
2827.4400 = \textit{area of the baſe.} \\
20 = \tfrac{2}{3} \textit{ of the height } (30). \\
\hline
56548.8000 = \textit{ſolidity required.}
\end{array}$$

2. In an hexagonal ſpherical dome, one ſide of the baſe is 20 feet; what is the ſolidity?

Anſ. 12000 *feet.*

PROBLEM IV.

To find the ſuperficial content of a ſpherical dome.

RULE.†

Multiply the area of the baſe by 2, and the product will be the ſuperficial content required.

---

* Domes and ſaloons are of various figures, but they are things that ſeldom occur in the practice of meaſuring.

† *The practical rule for elliptical domes is as follows:*

Rule. Add the height to half the diameter of the baſe, and this ſum multiplied by 1.5708 will give the ſuperficial content *nearly.*

EXAM-

EXAMPLES.

1. What will the painting an hexagonal ſpherical dome come to at 1*s.* per yard; each ſide of the baſe being 20 feet?

2.598076 = *area of a hexagon whoſe ſide is* 1.
400 = *ſquare of* 20.

1039.230400 = *area of the baſe.*
2

9)2078.460800 = *ſuperficial content required.*

2.0)230.940088

11.5470044 = 11*l.* 10*s.* 11*d.* *the expence of painting.*

PROBLEM V.

To find the ſolid content of a ſaloon.

RULE.*

1. Multiply the height of the arc, its projection, ¼ of the perimeter of the cieling, and 3.1416 continually together, and call the product A.

2. From

---

* *To find the ſuperficial content of a ſaloon.*

1. Find the area of the flat part of the cieling.

2. Find the convex ſurface of a cylinder, or cylindroid, whoſe length is equal to ¼ the perimeter of the cieling, and its diameters to twice the height and twice the projection of the arch.

3. Find the ſuperficial content of a dome of the ſame figure as the arch, and whoſe baſe is either a ſquare, or a figure ſimilar

2. From a ſide or diameter of the room take a like ſide or diameter of the cieling, and multiply the ſquare of the remainder by the proper factor (page 54) and this product again by $\frac{2}{3}$ of the height, and call the laſt product B.

3. Multiply the area of the flat cieling by the height of the arch, and this product added to the ſum of A and B will give the content required.

EXAMPLES.

1. What is the ſolid content of a ſaloon with a circular quadrantal arch of 2 feet radius, ſpringing over a rectangular room of 20 feet long and 16 feet wide?

Here the flat part of the cieling is 16 *feet by* 12; *and*

4)56

14 = $\frac{1}{4}$ *of the perimeter.*
2 = *height.*

28
2 = *projection.*

55

ſimilar to that of the cieling; the ſide being equal to the difference of a ſide of the room, and a ſide of the cieling.

4. Add theſe three articles together, and the ſum will give the ſuperficial content required.

Note. In a rectangular, circular, or polygonal room, the baſe of the dome will be a ſquare, a circle, or a like polygon.

3.1416

3.1416
56

188496
157080

175.9296=A.
20=*ſide of the room.*
16=*ſide of the cieling.*

4
4

16
1.000, &c.=*factor.*

16.000
$1\frac{1}{3}=\frac{2}{3}$ *of the height.*

16.000
5.333

21.333=B.
16
12

192=*area of the flat cieling.*
2=*height of the arch.*

384
175.9296
21.3333

581.2629=*ſolid content required.*

2. A cir-

2. A circular building of 40 feet diameter, and 25 feet high to the cieling, is covered with a ſaloon, whoſe circular arch is 5 feet radius; required the capacity of the room in cubic feet?

Anſ. 30779·45948 *feet.*

PROBLEM VI.

To find the ſolid content of the vacuity formed by a groin arch, either circular or elliptical.

RULE.

Multiply the area of the baſe by the height, and the product again by .904, and it will give the ſolidity required.

EXAMPLES.

1. What is the ſolid content of the vacuity formed by a circular groin, one ſide of its ſquare baſe being 12 feet?

12
12

144 = *area of the baſe.*
6 = *height.*

864
.904

3456
77760

781.056 = *ſolidity required.*

2. What is the ſolid content of the vacuity formed by an elliptical groin, one ſide of its ſquare baſe being 20 feet, and the height 6 feet? *Anſ.* 2169.6.

PROBLEM VII.

To find the concave ſuperficies of a circular groin.

RULE.*

Multiply the area of the baſe by 1.1416, and the product will be the ſuperficies required.

EXAMPLES.

1. What is the curve ſuperficies of a circular groin arch, one ſide of its ſquare being 12 feet?

12
12

144 = *area of the baſe.*
1.1416

45664
45664
11416

164.3904 = *ſuperficies required.*

---

* This rule may alſo be obſerved in elliptical groins, the error being too ſmall to be regarded in practice.

In meaſuring works where there are many groins in a range, the cylindrical pieces between the groins, and on their ſides, muſt be computed ſeparately.

2. What is the concave ſuperficies of a circular groin arch, one ſide of its ſquare being 9 feet?

Anſ. 92.4696.

And to find the ſolidity of the brick, or ſtone work, which forms the groin arches, obſerve the following

Rule. Multiply the area of the baſe by the height, including the work over the top of the groin, and this product leſſened by the ſolid content, found as before, will give the ſolidity required.

The general rule for meaſuring all arches is this:

From the content of the whole, conſidered as ſolid, from the ſpringing of the arch to the outſide of it, deduct the vacuity contained between the ſaid ſpringing and the under ſide of it, and the remainder will be the content of the ſolid part.

And becauſe the upper ſides of all arches, whether vaults or groins, are built up ſolid, above the haunces, to the ſame height with the crown, it is evident that the area of the baſe will be the whole content above-mentioned, taking for its thickneſs the height from the ſpringing to the top. And for the content of the vacuity to be deducted, take the area of its baſe, accounting its thickneſs to be $\frac{2}{3}$ of the greateſt inſide height. But it may be noted that the area uſed in the vacuity, is not exactly the ſame with that uſed in the ſolid; for the diameter of the former is twice the thickneſs of the arch leſs than that of the latter.

And although I have mentioned the deduction of the vacuity as common to both the vault and the groin, it is reaſonable to make it only in the former, on account of the waſte of materials and trouble to the workman, in cutting and fitting them for the angles and interſections.

Whoever wiſhes to ſee this ſubject more fully handled, may conſult *La Théorie et la Pratique de la Géométrie*, par *M. l'Abbé Deidier*; a work in which ſeveral parts of Menſuration and Practical Geometry are ſkilfully handled, the examples being moſtly wrought out in an eaſy familiar manner, and illuſtrated with obſervations, and figures very neatly executed.

OF

OF THE

CARPENTER'ſ RULE.

THIS inſtrument is commonly called *Cogeſhall*'s ſliding rule. It conſiſts of two pieces, of a foot in length each, which are connected together by means of a folding joint.

On one ſide of the rule, the whole length is divided into inches and half quarters, for the purpoſe of taking dimenſions. And on this face there are alſo ſeveral plain ſcales, divided by diagonal lines into twelfth parts, which are deſigned for planning ſuch dimenſions as are taken in feet and inches.

On one part of the other face there is a ſlider, and four lines marked A, B, C, and D; the two middle ones B and C being upon the ſlider.

Three of theſe lines A, B, C, are double ones, becauſe they proceed from one to 10 twice over; and the fourth line D is a ſingle one, proceeding from 4 to 40, and is called the *girt line*.

The uſe of the double lines, A and B, is for working proportions, and finding the areas of plane figures. And the uſe of the girt line D, and the other double line C, is for meaſuring ſolids.

When 1 at the beginning of any line is counted 1, then the 1 in the middle will be 10, and the 10 at the end 100. And when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000, &c. and all the ſmall diviſions are altered in value accordingly.

Upon the other part of this face there is a table of the value of a load of timber, at all prices, from *6d.* to *2s.* a foot.

Some rules have likewiſe a line of inches, or a foot divided decimally into 10th parts; as well as tables of board meaſure, timber meaſure, &c. but theſe will be beſt underſtood from a ſight of the inſtrument.

The uſe of the SLIDING RULE.

PROBLEM I.

To find the product of two numbers, as 7 and 26.

RULE.

Set 1 upon A, to one of the numbers (26) upon B; then againſt the other number (7) on A, will be found the product (182) upon B.

Note. If the third term runs beyond the end of the line, ſeek it on the other radius, or part of the line, and increaſe the product 10 times.

PROBLEM II.

To divide one number by another, as 510 by 12.

RULE.

Set the diviſor (12) on A, to 1 on B; then againſt the dividend (510) on A, is the quotient ($42\frac{1}{2}$) on B.

Note. If the dividend runs beyond the end of the line, diminiſh it 10 or 100 times to make it fall on A, and increaſe the quotient accordingly.

PRO-

PROBLEM III.

To ſquare any number, as 27.

RULE.

Set 1 upon D to 1 upon C; then againſt the number (27) upon D, will be found the ſquare (729) upon C.

If you would ſquare 270, reckon the 1 on D to be 100; and then the 1 on C will be 1000, and the product 72900.

PROBLEM IV.

To extract the ſquare root of any number, as 4268.

RULE.

Set 1 upon C, to 1 upon D; then againſt (4268) the number on C, is (65.3) the root on D.

To value this right you muſt ſuppoſe the 1 on C to be ſome of theſe ſquares 1, 100, 1000, &c. which is the neareſt to the given number, and then the root correſponding will be the value of the 1 upon D.

PROBLEM V.

To find a mean proportional between any two numbers, as 27 and 450.

RULE.

Set one of the numbers (27) on C, to the ſame on D; then againſt the other number (450) on C, will be the mean (112) on D.

Note. If one of the numbers overruns the line, take the 100th part of it, and augment the anſwer 10 times.

PROBLEM VI.

Three numbers being given, to find a fourth proportional; ſuppoſe 12, 28, *and* 57.

RULE.*

Set the firſt number (12) upon A, to the ſecond (28) upon B; then againſt the third number (57) on A, is the fourth (133) on B.

Note. If one of the middle numbers runs off the line, take the 10th part of it only, and augment the anſwer 10 times.

The finding a third proportional is exactly the ſame, the ſecond number being twice repeated.

Thus, ſuppoſe a third proportional was required to 21 and 32.

Set the firſt 21 on B, to the ſecond 32 on A; then againſt the ſecond 32 on B, is 48.8 on A, which is the third proportional required.

---

* The uſe of the rule in board and timber meaſure will be ſhewn in what follows.

If the breadth of a board be given; to find how much in length will make a ſquare foot.

Rule. If the board be narrow, it will be found in the table of board meaſure on the rule; but, if not, ſhut the rule, and ſeek the breadth in the line of board meaſure, running along the rule, from that table; then over againſt it, on the oppoſite ſide, is the length in inches required.

The ſide of the ſquare of a piece of timber being given; to find how much in length will make a foot ſolid.

RULE. If the timber be ſmall, it will be found in the table of timber meaſure on the rule; but, if not, look for the ſide of the ſquare, in the line of timber meaſure, running along the rule, from that table, and againſt it in the line of inches is the length required.

OF
TIMBER MEASURE.

PROBLEM I.

To find the area, or ſuperficial content, of a board or plank.

RULE.

MULTIPLY the length by the breadth, and the product will be the content required.

Note. When the board is tapering, add the breadths of the two ends together, and take ½ the ſum for the mean breadth.

By the SLIDING RULE.

Set 12 on B to the breadth in inches on A, then againſt the length in feet on B, is the content on A, in feet and fractional parts, as required.

EXAMPLES.

1. What is the value of a plank, whoſe length is 8 feet 6 inches, and breadth 1 foot 3 inches throughout, at 2½*d.* per foot?

feet	*in.*	
8	6	
1	3	
8	6	
2	1	6
10	7	6 *the content.*

	10	7 6
2*d.* *is* $\frac{1}{6}$	1	8
$\frac{1}{2}$ *is* $\frac{1}{4}$		5
in. 6 *is* $\frac{1}{2}$		1¼
in. 1 *is* $\frac{1}{6}$		¼
	2*s.*	2½*d. the anſwer.*

By the SLIDING RULE.

As 12 *on B* : 15 *on A* :: 8½ *on B* : 10½ *on A.*

2. What is the content of a board, whoſe length is 5 feet 7 inches, and breadth 1 foot 10 inches?

feet in. pa.
Anſ. 10 2 10

3. At 1½*d.* per foot, what is the value of a plank, whoſe length is 12 feet 6 inches, and breadth 11 inches throughout? *Anſ.* 1*s.* 5*d.*

4. Find the value of 5 oaken planks at 3*d.* per foot, each being 17½ feet long, and their particular breadths as follows: viz. two of 13½ inches in the middle, one of 14½ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and 11¼ at the narrower? *Anſ.* 1*l.* 5*s.* 9¼*d.*

PROBLEM II.

To find the ſolidity of ſquared or four-ſided timber.

RULE.

RULE.*

Multiply the mean breadth by the mean thickneſs, and this product again by the length, and it will give the ſolidity required.

By the SLIDING RULE.

As the length in feet on *C* : 22 on *D* : : quarter girt in inches on *D* : ſolidity on *C*.

---

* *Note* 1. If the ſtick be equally broad and thick throughout, the breadth and thickneſs, any where taken, will be the mean breadth and thickneſs.

2. If the tree tapers regularly from one end to the other, the breadth and thickneſs, taken in the middle, will be the mean breadth and thickneſs.

3. If the ſtick does not taper regularly, but is thicker in ſome places than in others, let ſeveral different dimenſions be taken, and their ſum divided by the number of them will give the mean dimenſions.

This method of finding the mean dimenſions is moſtly uſed in practice, but, in many caſes, it is exceedingly erroneous.

The quarter girt, likewiſe, which is mentioned in the proportion by the ſliding rule, is ſubject to error. It is not the fourth part of the circumference, but the ſquare root of the product ariſing from multiplying the mean breadth by the mean thickneſs.

In order to ſhew the fallacy of taking ¼ of the girt for the ſide of a mean ſquare, take the following example:

Suppoſe a piece of timber to be 24 feet long, and a foot ſquare throughout, and let it be ſlit into two equal parts, from end to end.

Then the ſum of the ſolidities of the two parts, by the quarter girt method, will be 27 feet, but the true ſolidity is 24 feet; and if the two pieces were very unequal, the difference would be ſtill greater.

EXAMPLES.

1. The length of a piece of timber is 20½ feet, the breadth at the greater end is 1 foot 9 inches, and the thickneſs 1 foot 3 inches; and at the leſſer end the breadth is 1 foot 6 inches, and the thickneſs 1 foot: what is the ſolidity?

1.75 = *greater breadth.*
1.5 = *leſſer breadth.*

2)3.25

1.625 = *mean breadth.*

1.25 = *greater thickneſs.*
1.00 = *leſſer thickneſs.*

2)2.25

1.125 = *mean thickneſs.*

By Decimals.

```
   1.625
   1.125
  ------
    8125
   3250
  1625
 1625
 -------
1.828125
    20.5
 -------
  9140625
 36562500
 ----------
37.4765625
```

37.4765625 = *content.*

By Croſs Multiplication.

```
fe. in. pa.
 1   7   6
 1   1   6
------------
 1   7   6
     1   7   6
         9   9
------------
 1   9  11   3
20   6
------------
36   6   9   0
    10  11   7   6
----------------
37   5   8   7   6=content.
```

By the SLIDING RULE.

As 1 *upon* B : $19\frac{6}{12}$ *upon* A : : $13\frac{6}{12}$ *upon* B : $263\frac{25}{100}$ *upon* A, *the mean ſquare.*

As 16 *upon* C : 4 *upon* D : : 1.8 *upon* C : 16.2 *upon* D, *the ſide of the mean ſquare.*

As $20\frac{1}{2}$ *upon* C : 12 *upon* D : : 16.2 *upon* D : $37\frac{5}{12}$ *upon* C, *the anſwer.*

2. The length of a piece of timber is 24.5 feet, and its ends are equal ſquares, whoſe ſides are each 1.04 feet: what is the ſolidity? *Anſ.* 26 *feet* 6 *inches.*

3. The length of a piece of timber is 20.38 feet, and the ends are unequal ſquares, the ſide of the greater being $19\frac{1}{8}$ inches, and tnat of the leſſer $9\frac{1}{8}$ inches: what is the ſolidity? *Anſ.* 29 *feet* 4 *inches.*

4. The length of a piece of timber is 27.36 feet; at the greater end the breadth is 1.78 feet, and the thickneſs 1.23 feet; and at the leſſer end the breadth is 1.04 feet, and the thickneſs .91 feet: what is the ſolidity? *Anſ.* 41.726 *feet.*

PROBLEM III.

To find the ſolidity of round or unſquared timber.

RULE I*.

Multiply the ſquare of the quarter girt (or $\frac{1}{4}$ of the circumference) by the length, and the product will be the content, *according to the common practice.*

By the SLIDING RULE.

As the length upon C : 12 upon D : : $\frac{1}{4}$ girt upon D : content upon C.

EXAMPLES.

1. A piece of timber is $9\frac{1}{4}$ feet long, and the quarter girt is 39 inches: what is the ſolidity?

---

* Let c = girt or circumference, and l = length of the tree.

Then $\frac{c}{4} \times \frac{c}{4} \times l = \frac{c^2 l}{16}$ = content of the tree according to the rule.

And $\frac{c^2}{4 \times 3.1416} \times l = \frac{c^2 l}{12.5664}$ = true content, according to the rule for finding the content of a cylinder.

But $\frac{c^2 l}{12.5664}$ differs from $\frac{c^2 l}{16}$ by nearly $\frac{1}{4}$ part of the whole, and therefore the rule is exceedingly erroneous.

When the tree is tapering, the mean girt is found in the ſame manner as in board meaſure. Or if the tree be very irregular, the beſt way is to divide it into a certain number of lengths, and find the content of each part ſeparately.

When trees have their bark on, an allowance is generally made, by deducting ſo much from the girt as is judged ſufficient to reduce it to ſuch a circumference as it would have without its bark. In oak this allowance is about $\frac{1}{10}$ or $\frac{1}{12}$ part of the girt; but for elm, beech, aſh, &c. whoſe bark is not ſo thick, the deduction ought to be leſs.

By Decimals.

3.25=39 *inches.*
3.25
1625
650
975
10.5625
9.75=9¾ *feet.*
528125
739375
950625
102.984375=*solidity.*

By Cross Multiplication.

f. in.
3 3=39 *inches.*
3 3
9 9
9 9
10 6 9
9 9 =9¾ *feet.*
95 0 9
7 11 0 9
102 11 9 9=*solidity.*

By the Sliding Rule.

As 9¾ *upon* C : 12 *upon* D : : 39 *upon* D : 103 *upon* C, *the content.*

2. The length of a tree is 25 feet, and the girt throughout $2\frac{1}{2}$ feet: what is its ſolidity?

Anſ. 9 *feet* 9 *inches.*

3. The length of a tree is $14\frac{1}{2}$ feet, and its girt in the middle 3.15 feet: required the ſolidity?

Anſ. 9 *feet nearly.*

4. The girts of a tree in 4 different places are as follows: in the firſt place 5 feet 9 inches, in the ſecond 4 feet 5 inches, in the third 4 feet 9 inches, and in the fourth 3 feet 9 inches; and the length of the whole tree is 15 feet: what is the ſolidity?

Anſ. 20 *feet* 5 *inches.*

5. An oak tree is 45 feet 7 inches long, and its quarter girt 3 feet 8 inches; what is the ſolid content, allowing $\frac{1}{12}$ for the bark? *Anſ.* 515 *feet, nearly.*

RULE II.*

Multiply the ſquare of $\frac{1}{5}$ of the girt by twice the length, and the product will be the ſolidity, *extremely near the truth.*

By

---

* Let c = circumference, and l = length, as before.

Then $\frac{c}{5} \times \frac{c}{5} \times 2l = \frac{2c^2l}{25} = \frac{c^2l}{12.5} =$ content of the tree according to the rule.

And the true content is $= \frac{c^2l}{12.5664}$, as was before ſhown.

But $\frac{c^2l}{12.5}$ differs from $\frac{c^2l}{12.5664}$ by only about $\frac{1}{190}$ part of the whole, and is therefore ſufficiently near the truth for any practical purpoſe.

This rule is full as eaſy in practice as the falſe one, and therefore ought to be generally uſed, ſince the eaſe of the other method is the only argument which is alledged for employing it.

The

By the SLIDING RULE.

At twice the length upon C : 12 upon D :: $\frac{1}{5}$ of the girt upon D : content upon C.

EXAM-

The following rule was given me by *Mr. Barrow*, and is a ſtill nearer approximation.

Rule. Multiply the ſquare of the circumference by the length, and take $\frac{1}{11}$ of the product; from this laſt number ſubtract $\frac{1}{8}$ of itſelf, and the remainder will be the anſwer.

For $\frac{c^2 l}{12.5664} = \frac{7}{88}$ of $c^2 l$ *very nearly*, $= \frac{8}{88} - \frac{1}{88} \times c^2 l = \frac{c^2 l}{11} - \frac{1}{8}$ of $\frac{c^2 l}{11}$, which is the ſame as the rule; and differs from the truth by only 1 foot in 2300.

The following problems, as well as many other things in this ſection, were taken from *Dr. Hutton*'s Menſuration. They are given to ſhew the artifices that may be uſed in meaſuring timber according to the falſe method now practiſed, and the abſolute neceſſity there is of aboliſhing it.

PROBLEM I.

To find where a tree muſt be cut, ſo that the two parts, meaſured ſeparately, ſhall produce a greater ſolidity than that of the whole tree, or any other two parts of it.

RULE. Cut it through exactly in the middle, or at $\frac{1}{2}$ of the length, and the two parts will meaſure the moſt poſſible.

Demon. Put G = greateſt girt, g = leaſt, x = girt at the ſection, L = whole length, and z = length to be cut off the leſs end.

Then, by ſimilar figures, $L : z :: G - g : x - g$; or $x = \frac{Gz - gz}{L} + g$.

But $\overline{g + x}|^2 \times z + \overline{G + x}|^2 \times \overline{L - z}$ = a maximum.

And if the fluxion of this expreſſion be put equal to nothing, and the value of x be ſubſtituted inſtead of it, there will reſult $z = \frac{1}{2} L$. Q. E. D.

Example.

EXAMPLES.

1. A piece of timber is $9\frac{1}{4}$ feet long, and $\frac{1}{5}$ of the girt is 2.6 feet: what is the ſolidity?

By

Example. Suppoſing a tree to girt 14 feet at the greater end, 2 feet at the leſs, and 8 feet in the middle, and that the length is 32 feet.

Then, by the common method, the whole tree meaſures only 128.

But, when cut through the middle, the greater part meaſures 121 feet, and the leſs part 25 feet.

And the ſum of theſe two parts is 146 feet, which exceeds the whole by 18 feet, and is the moſt that it can be made to meaſure by cutting it into two parts.

PROBLEM II.

To find where a tree muſt be cut, ſo that the part next the greater end may meaſure the moſt poſſible.

Rule. Cut it where the girt is $\frac{1}{3}$ of the greateſt girt, and the greater end will meaſure to the moſt poſſible.

Demon. By uſing the ſame notation as in the laſt problem, we ſhall have $x = \frac{Gz - gz}{L} + g$, and $\overline{G + x)} \times \overline{L - z} =$ a maximum.

And, if this be put into fluxions, it will give $z = \frac{G - 3g}{G - g} \times \frac{1}{3}L$, and $x = \frac{G - g}{L} \times z + g = \frac{1}{3}G$. Q. E. D.

Example. Taking here the ſame example as before, we ſhall have $12 : 8 :: \frac{32}{3} : 7\frac{1}{9} =$ length to be cut off, $24\frac{8}{9} =$ length of the remaining part, and $4\frac{2}{3} =$ girt at the ſection.

But the content of the whole tree is only 128 feet, and the content of the greater part $135\frac{4}{8}\frac{1}{1}$ feet, which exceeds 128 by $7\frac{4}{8}\frac{1}{1}$, and is the greateſt poſſible.

Note.

By Decimals.

$$\begin{array}{r}
2.6\\
2.6\\
\hline
156\\
52\\
\hline
6.76\\
9.75\\
\hline
3380\\
4732\\
6084\\
\hline
65.9100\\
2\\
\hline
131.8200 = \textit{content.}
\end{array}$$

By

Note. If the greateſt girt does not exceed 3 times the leaſt, the tree cannot be cut as is required by the problem ; for when the leaſt girt is exactly equal to $\frac{1}{3}$ of the greater, the tree already meaſures to the moſt poſſible.

PROBLEM III.

To cut a tree, ſo that the part next the greater end may meaſure exactly the ſame as the whole tree.

Rule 1. Call the ſum of the girts of the two ends s, and their difference d.

2. Multiply d by the ſum of d and $4s$, and from the ſquare root of the product take the difference between d and $2s$.

3. Then as $2d$ is to this remainder, ſo is the whole length of the tree, to the length to be cut off the ſmaller end.

Demon.

By the SLIDING RULE.

As 19.15 *upon* C : 12 *upon* D :: 31⅕ *in. upon* D : 132, *the content upon* C.

2. If the length of a tree be 24 feet, and the girt throughout 8 feet: what is the content?

Anſ. 123 *feet, nearly.*

3. If a tree girt 14 feet at the thicker end, and 2 feet at the ſmaller end: required the ſolidity when the length is 24 feet? *Anſ.* 11½ *feet, nearly.*

4. A tree girts in five different places as follows: in the firſt place 9.43 feet, in the ſecond 7.92 feet, in the third 6.15 feet, in the fourth 4.74 feet, and in the fifth 3.16 feet; and the whole length is 17¼ feet: what is the ſolidity? *Anſ.* 54.4065 *feet.*

Demon. Uſing ſtill the ſame notation, we ſhall have $s^2 L = \overline{L - z} \times \overline{G + x}|^2$; and by ſubſtituting for x its value $\frac{dz}{L} + g$, it will be $z = \frac{L}{2d} \times \overline{\sqrt{\overline{\overline{4s + d} \times d}} - 2s + d}$, and $x = \frac{1}{2} \sqrt{\overline{\overline{4s + d} \times d}} - s$. Q. E. D.

The rule may be illuſtrated by taking the ſame example as in the laſt problems.

Thus, ſince $s = 16$, $d = 12$, and the length $L = 32$, $\frac{32}{24} \times \overline{\sqrt{\overline{76 \times 12}} - 20} = \frac{16}{3} \sqrt{57} - 26\frac{2}{3} = 13.599118 =$ length to be cut off; and therefore the length of the remaining part is 18.400882.

Alſo, $\frac{1}{2} \times \overline{\sqrt{\overline{76 \times 12}} - 8} = 2\sqrt{57} - 8 = 7.099669 =$ girt at the ſection. Whence the girt in the middle of the greater part is $\frac{14 + 7.099669}{2} = 10.549834$, whoſe ¼ part is 2.637458, and conſequently the content of this part is $\overline{2.637458}|^2 \times 18.400882 = 128$ the ſame as the content of the whole tree.

OF

OF

SPECIFIC GRAVITY.

THE ſpecific gravities of bodies are their relative weights contained under the ſame given magnitude, as a cubic foot, a cubic inch, &c.

The ſpecific gravities of ſeveral ſorts of bodies are expreſſed by the numbers annexed to their names in the following table.

A table of the ſpecific gravities of bodies.

Fine gold	19640	Brick	2000
Standard gold	18888	Light earth	1984
Quickſilver	14000	Solid gunpowder	1745
Lead	11325	Sand	1520
Fine ſilver	11091	Pitch	1150
Standard ſilver	10535	Dry box wood	1030
Copper	9000	Sea water	1030
Gun metal	8784	Common water	1000
Caſt braſs	8000	Dry oak	925
Steel	7850	Gunpowder, ſhaken	922
Iron	7645	Dry aſh	800
Caſt iron	7425	Dry maple	755
Tin	7320	Dry elm	600
Marble	2700	Dry fir	550
Common ſtone	2520	Cork	240
Loom	2160	Air	1¼

Note. As a cubic foot of water weighs juſt 1000 ounces Avoirdupois, the numbers in this table expreſs not only the ſpecific gravities of the ſeveral bodies, but alſo the weight of a cubic foot of each, in Avoirdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be readily known.

PROBLEM I.

To find the magnitude of a body from its weight being given.

RULE.

As the tabular ſpecific gravity of the body, is to its weight in Avoirdupois ounces,

So is one cubic foot, or 1728 cubic inches, to its content in feet, or inches, reſpectively.

EXAMPLES.

1. Required the content of an irregular block of common ſtone which weighs 1 *cwt.* or 112 *lbs.*?

112 *lbs.*
16
672
112
2520 : 1792 :: 1728
1728
14336
3584
12544
1792
2520)3096576(1228$\frac{4}{5}$ *cubic inches the anſ.*
252
576
504
725
504
2217
2016
2016

2. How many cubic inches of gunpowder are there in one pound weight? *Anſ.* 30 *nearly.*

3. How many cubic feet are there in a ton weight of dry oak? *Anſ.* $38\frac{138}{185}$.

PROBLEM II.

To find the weight of a body from its magnitude being given.

RULE.

As one cubic foot, or 1728 cubic inches, is to the content of the body,

So is its tabular ſpecific gravity, to the weight of the body.

EXAMPLES.

1. Required the weight of a block of marble, whoſe length is 63 feet, and its breadth and thickneſs each 12 feet; theſe being the dimenſions of one of the ſtones in the walls of Balbec.

```
                  63
                  12
                 ———
                 756
                  12
                 ———
      1  :      9072  :: 2700
                2700
             ———————
             6350400
            18144
         ———————————
16{ 4  |    24494400 }
   { 4  |     6123600 } oz.
   112  |     1530900 lbs.
    20  |       13668 cwts.
                  683 ton.
```

Anſ. 683$\frac{2}{5}$ *tons, which is equal to the burthen of an Eaſt India ſhip.*

2. What is the weight of a pint of gunpowder ale-meaſure? *Anſ.* 19 *oz. nearly.*

3. What is the weight of a block of dry oak, which meaſures ten feet in length, 3 feet in breadth, and $2\frac{1}{4}$ feet deep? *Anſ.* $4335\frac{15}{16}$ *lb.*

PROBLEM III.

To find the ſpecific gravity of a body.

RULE.

Caſe 1. When the body is heavier than water, weigh it both in water and out of water, and the difference will be the weight loſt in the water.

Then, as the weight loſt in water, is to the whole weight,

So is the ſpecific gravity of water, to the ſpecific gravity of the body.

EXAMPLE.

A piece of ſtone weighed in air 10 pounds, but in water only $6\frac{3}{4}$ lb. Required its ſpecific gravity?

10
$6\frac{3}{4}$

$3\frac{1}{4} : 10 :: 1000 :$
$13 : 40 :: 1000 :$
40

$13)40000(3077$
39

100
91

90

Caſe 2. When the body is lighter than water, ſo that it will not quite ſink; affix to it another body heavier than water, ſo that the maſs compounded of the two may ſink together.

Weigh the heavier body and the compound maſs ſeparately both in water and out of it, and find how much each loſes in water, by ſubtracting its weight in water from its weight in air.

Then as the difference of theſe remainders is to the weight of the light body in air,

So is the ſpecific gravity of water to the ſpecific gravity of the body.

EXAMPLE.

Suppoſe a piece of elm weighs in air 15lb. and that a piece of copper which weighs 18lb. in air, and 16lb. in water, is affixed to it, and that the compound weighs 6lb. in water; Required the ſpecific gravity of the elm?

18 *in air* 33
16 *in water* 6
— —
loſs 2 27 *loſs*
2
—
25

25 : 15 :: 1000 : 600 *Anſ.*

PROBLEM IV.

To find the quantities of two ingredients in a given compound.

RULE.

RULE.

Take the differences of every pair of the three ſpecific gravities, *viz.* of the compound and each ingredient, and multiply the difference of every two by the third.

Then as the greateſt product is to the whole weight of the compound, ſo is each of the other products to the weights of the two ingredients.

EXAMPLE.

A compoſition of 112lb. being made of tin and copper, whoſe ſpecific gravity is found to be 8784; Required the quantity of each ingredient? The ſpecific gravity of tin being 7320, and of copper 9000.

9000	9000	8754
7320	8784	7320
1680	216	1464 *diff.*
8784	7320	9000
702720	4320	13176000
52704	648	
8784	1512	
14757120	1581120	

14757120 : 112 :: 13176000
112

26352000
13176
13176

14757120)1475712000(100

Anſ. 100*lb. of copper* } *in the compoſition.*
and 12*lb. of tin* }

MISCELLANEOUS QUESTIONS.

1. WHAT difference is there between a floor 48 feet long, and 30 feet broad, and two others each of half the dimenſions? *Anſ.* 720 *feet.*

3. From a mahogany plank 26 inches broad, a yard and a half is to be ſawed off; what diſtance from the end muſt the line be ſtruck? *Anſ.* 6.23 *feet.*

3. A joiſt is 8½ inches deep, and 3½ broad: what will be the dimenſions of a ſcantling juſt as big again as the joiſt, that is 4¼ inches broad?

Anſ. 12.52 *inches deep.*

4. A roof is 24 feet 8 inches by 14 feet 6 inches, and is to be covered with lead at 8*lbs.* to the foot; what will it come to at 18*s.* per cwt?

Anſ. 22*l.* 19*s.* 10¼*d.*

5. What is the ſide of that equilateral triangle, whoſe area coſt as much paving at 8*d.* per foot, as the palliſading the three ſides did at a guinea per yard?

Anſ. 72.746 *feet.*

6. The two ſides of an obtuſe-angled triangle are 20 and 40 poles: what muſt the length of the third ſide be that the triangle may contain juſt an acre?

Anſ. 58.876, *or* 23.099.

7. If two ſides of a triangle, whoſe area is $60\sqrt{3}$, be 12 and 20: what is the third ſide? *Anſ.* 28.

8. If an area of 24 be cut off from a triangle, whoſe three ſides are 13, 14, and 15, by a line parallel to the longeſt ſide: what are the lengths of the ſides including that area?

Anſ. $\frac{13}{7}\sqrt{14}$, $2\sqrt{14}$, *and* $\frac{15}{7}\sqrt{14}$.

9. The diſtance of the centres of two circles, whoſe diameters are each 50, is equal to 30: what is the area of the ſpace incloſed by their circumferences?

Anſ. 559.119.

10. The area of an equilateral triangle, whoſe baſe falls on the diameter, and its vertex in the middle of the arc of a ſemi-circle, is equal to 100: what is the diameter of the ſemi-circle?

Anſ. 26.3214.

11. The four ſides of a field, whoſe diagonals are equal to each other, are 25, 35, 31, and 19 poles, reſpectively: what is the area?

Anſ. 4 *ac.* 1 *ro.* 38 *poles.*

12. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whoſe diameter is 289?

Anſ. 278.6716.

13. A cable which is 3 feet long, and 9 inches in compaſs, weighs 22*lbs.* what will a fathom of that cable weigh whoſe diameter is 9 inches?

Anſ. 434.26*lbs.*

14. A circular fiſh-pond is to be dug in a garden, that ſhall take up juſt half an acre, what muſt the length of the chord be that ſtrikes the circle?

Anſ. 27.75 *yards.*

15. A carpenter is to put an oaken curb to a round well, at 8*d.* per foot ſquare; the breadth of the curb is to be 7$\frac{1}{2}$ inches, and the diameter within 3$\frac{1}{2}$ feet: what will be the expence? *Anſ.* 5*s.* 2$\frac{1}{4}$*d.*

16. Suppoſe the expence of paving a ſemi circular plot, at 2*s.* 4*d.* per foot, amounted to 10*l.* what is the diameter of it? *Anſ.* 14.7737.

* 17. Seven men bought a grinding-ſtone of 60 inches in diameter, each paying $\frac{1}{7}$ part of the expence; what part of the diameter muſt each grind

down for his ſhare? *Anſ. The 1ſt.* 4.4508, *2d.* 4.8400, *3d.* 5.3535, *4th.* 6.0765, *5th.* 7.2079, *6th.* 9.3935, *and the 7th.* 22.6778.

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* For an excellent geometrical conſtruction of this queſtion, ſee *Dr. Hutton*'s Diarian Miſcellany, page 54.

18. A gentleman has a garden 100 feet long, and 80 feet broad, and a gravel walk is to be made of an equal width half round it: what muſt the width of the walk be ſo as to take up juſt half the ground? *Anſ.* 25.968 *feet.*

19. In the midſt of a meadow well ſtored with
graſs,
I took juſt an acre to tether my aſs;
How long muſt the cord be, that feeding all round,
He may'nt graze leſs or more than an acre of
ground? *Anſ.* 39.25073 *yards.*

20. A maltſter has a kiln that is 16 feet 6 inches ſquare; now he wants to pull it down, and build a new one that will dry three times as much at a time as the old one did: what muſt be the length of its ſide? *Anſ.* 28 *feet* 7 *inches.*

21. If a round ciſtern be 26.3 inches diameter, and 52.5 inches deep: how many inches diameter muſt a ciſtern be to hold twice the quantity, the depth being the ſame? *Anſ.* 37.19 *inches.*

22. A May-pole, whoſe top was broken off by a blaſt of wind, ſtruck the ground at 15 feet diſtance from the top of the pole: what was the height of the whole May-pole, ſuppoſing the length of the broken piece to be 39 feet? *Anſ.* 75 *feet.*

23. What will the diameter of a globe be, when the ſolidity and ſuperficial content thereof are equal to each other? *Anſ.* 6.

24. How many three inch cubes can be cut out of a 12 inch cube? *Anſ.* 64.

25. A farmer borrowed part of a hay-rick of his neighbour, which meaſured 6 feet every way, and paid him back again by two equal cubical pieces,

each of whoſe ſides were three feet: Query, whether the lender was fully paid?

Anſ. He was paid ¼ part only.

26. What will the painting a conical church-ſpire come to at 8*d.* per yard; ſuppoſing the circumference of the baſe to be 64 feet, and the altitude 118 feet?

Anſ. 14*l.* 0*s.* 8¾*d.*

27. What will a marble fruſtum of a cone come to at 12*s.* per ſolid foot; the diameter of the greater end being 4 feet, that of the leſſer end 1½ feet, and the length of the ſlant ſide 8 feet? *Anſ.* 30*l.* 1*s.* 10*d.*

28. The diameter of a legal Wincheſter buſhel is 18½ inches, and its depth 8 inches: what muſt the diameter of that buſhel be whoſe depth is 7½ inches?

Anſ. 19.10671.

29. Three men bought a tapering piece of timber, which was the fruſtum of a ſquare pyramid; one ſide of the greater end was 3 feet, one ſide of the leſſer end 1 foot, and the length 18 feet: what is the length of each man's piece, ſuppoſing they paid equally, and are to have equal ſhares? *Anſ. 1ſt.* 3.269, *2d.* 4.559, *and the 3d.* 10.172, *reckoning from the greater end to the leſs.*

30. Suppoſe the ball at the top of St. Paul's Church is 6 feet in diameter: what did the gilding of it come to at 3¼*d.* per ſquare inch?

Anſ. 237*l.* 1*s.* 10¼*d.*

31. A perſon wants a cylindric veſſel of 3 feet deep, that ſhall hold twice as much as a veſſel of 28 inches deep, and 46 inches in diameter: what muſt be the diameter of the veſſel required?

Anſ. 57.37 *inches.*

32. Two porters agreed to drink off a quart of ſtrong beer between them, at two pulls, or a draught each; now, the firſt having given it a black eye, as it is called, or drank till the ſurface of the liquor touched the oppoſite edge of the bottom, gave the

remaining part of it to the other: what was the difference of their ſhares, ſuppoſing the pot was the fruſtum of a cone, the depth being 5.7 inches, the diameter at the top 3.7 inches, and that of the bottom 4.23 inches? *Anſ.* 7.07 *cubic inches.*

33. Three perſons having bought a ſugar-loaf, want to divide it equally amongſt them by ſections parallel to the baſe; it is required to find the altitude of each perſon's ſhare, ſuppoſing the loaf to be a cone, whoſe height is 20 inches? *Anſ.* 13.867 *the upper part,* 3.604 *the middle part, and* 2.528 *the lower part.*

34. How high above the ſurface of the earth muſt a perſon be raiſed to ſee $\frac{1}{3}$ of its ſurface?

Anſ. To the height of the earth's diameter.

35. A cubical foot of braſs is to be drawn into a wire of $\frac{1}{40}$ of an inch in diameter; what will be the length of the wire, allowing no leſs in the metal?

Anſ. 97784.797 *yards, or near* 56 *miles.*

35. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raiſe one foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth muſt the ditch be dug, ſuppoſing its breadth to be every where 8 feet? *Anſ.* $7\frac{13}{288}$ *feet.*

36. Of what diameter muſt the bore of a cannon be, which is caſt for a ball of 24 lbs. weight, ſo that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? *Anſ.* 5.757 *inches.*

37. At what height from the bottom muſt an upright cone be cut, ſo that the greateſt cylinder poſſible may be formed from the lower part of it?

Anſ. At $\frac{1}{3}$ *of the height.*

38. The ellipſe in Groſvenor ſquare, meaſures 840 links acroſs the longeſt way, and 612 the ſhorteſt, within the rails: now the walls being 14 inches thick, it is required to find what ground they incloſe, and

what they ſtand upon? *Anſ. They incloſe* 4 *ac.* 0 *ro.* 6 *po. and ſtand on* $1760\frac{1}{2}$ *ſquare feet.*

49. If a heavy ſphere whoſe diameter is 4 inches, be put into a conical glaſs, full of water, whoſe diameter is 5, and altitude 6 inches; it is required to find how much water will run over?

Anſ. $\frac{35}{47}$ *of a pint nearly.*

40. Suppoſe it be found, by meaſurement, that a man of war, with its ordnance, rigging and appointments, draws ſo much water as to diſplace 50000 cubic feet of water: required the weight of the veſſel:

Anſ. $1395\frac{1}{16}$ *tons.*

41. One ev'ning I chanc'd with a Tinker to ſit,
Whoſe tongue ran a great deal too faſt for his wit;
He talk'd of his art with abundance of mettle;
So I aſk'd him to make me a flat-bottom'd kettle:
Let the top and the bottom diameters be,
In juſt ſuch proportion as five is to three:
Twelve inches the depth I propos'd, and no more;
And to hold in ale gallons ſeven leſs than a ſcore.
He promis'd to do it, and ſtrait to work went;
But when he had done it he found it too ſcant.
He alter'd it then, but too big he had made it;
For though it held right, the diameters fail'd it:
Thus making it often too big and too little,
The Tinker at laſt had quite ſpoilt his kettle;
But he ſwears he will bring his ſaid promiſe to paſs,
Or elſe that he'll ſpoil every ounce of his braſs.
Now, to keep him from ruin, I pray find him out
The diameter's length, for he'll ne'er do it I doubt.

Anſ. The bottom diameter is 14.64017, *and the top diameter* 24.40028.

A
TABLE
OF THE
AREAS OF THE SEGMENTS OF A CIRCLE,

Whoſe Diameter is Unity, and ſuppoſed to be divided into 1000 equal Parts.

Verſed Sine.	*Seg. Area.*	Verſed Sine.	*Seg. Area.*	Verſed Sine.	*Seg. Area.*
.001	.000042	.024	.004921	.047	.013392
.002	.000119	.025	.005230	.048	.013818
.003	.000219	.026	.005546	.049	.014247
.004	.000337	.027	.005867	.050	.014681
.005	.000470	.028	.006194	.051	.015119
.006	.000618	.029	.006527	.052	.015561
.007	.000779	.030	.006865	.053	.016007
.008	.000951	.031	.007209	.054	.016457
.009	.001135	.032	.007558	.055	.016911
.010	.001329	.033	.007913	.056	.017369
.011	.001533	.034	.008273	.057	.017831
.012	.001746	.035	.008638	.058	.018296
.013	.001968	.036	.009008	.059	.018766
.014	.002199	.037	.009383	.060	.019239
.015	.002438	.038	.009763	.061	.019716
.016	.002685	.039	.010148	.062	.020196
.017	.002940	.040	.010537	.063	.020680
.018	.003202	.041	.010931	.064	.021168
.019	.003471	.042	.011330	.065	.021659
.020	.003748	.043	.011734	.066	.022154
.021	.004031	.044	.012142	.067	.022652
.022	.004322	.045	.012554	.068	.023154
.023	.004618	.046	.012971	.069	.023659

The

The Areas of the Segments of a Circle.

Ver-ſed Sine.	*Seg. Area.*	Ver-ſed Sine.	*Seg. Area.*	Ver-ſed Sine.	*Seg. Area*
.070	.024168	.103	.042687	.136	.064074
.071	.024680	.104	.043296	.137	.064760
.072	.025195	.105	.043908	.138	.065449
.073	.025714	.106	.044522	.139	.066140
.074	.026236	.107	.045139	.140	.066833
.075	.026761	.108	.045759	.141	.067528
.076	.027289	.109	.046381	.142	.068225
.077	.027821	.110	.047005	.143	.068924
.078	.028356	.111	.047632	.144	.069625
.079	.028894	.112	.048262	.145	.070328
.080	.029435	.113	.048894	.146	.071033
.081	.029979	.114	.049528	.147	.071741
.082	.030526	.115	.050165	.148	.072450
.083	.031076	.116	.050804	.149	.073161
.084	.031629	.117	.051446	.150	.073874
.085	.032186	.118	.052090	.151	.074589
.086	.032745	.119	.052736	.152	.075306
.087	.033307	.120	.053385	.153	.076026
.088	.033872	.121	.054036	.154	.076747
.089	.034441	.122	.054689	.155	.077469
.090	.035011	.123	.055345	.156	.078194
.091	.035585	.124	.056003	.157	.078921
.092	.036162	.125	.056663	.158	.079649
.093	.036741	.126	.057326	.159	.080380
.094	.037323	.127	.057991	.160	.081112
.095	.037909	.128	.058658	.161	.081846
.096	.038496	.129	.059327	.162	.082582
.097	.039087	.130	.059999	.163	.083320
.098	.039680	.131	.060672	.164	.084059
.099	.040276	.132	.061348	.165	.084801
.100	.040875	.133	.062026	.166	.085544
.101	.041476	.134	.062707	.167	.086289
.102	.042080	.135	.063389	.168	.087036

The Areas of the Segments of a Circle.

Ver-ſed Sine.	*Seg. Area.*	Ver-ſed Sine.	*Seg. Area.*	Ver-ſed Sine.	*Seg. Area.*
.169	.087785	.202	.113426	.235	.140688
.170	.088535	.203	.114230	.236	.141537
.171	.089287	.204	.115035	.237	.142387
.172	.090041	.205	.115842	.238	.143238
.173	.090797	.206	.116650	.239	.144091
.174	.091554	.207	.117460	.240	.144944
.175	.092313	.208	.118271	.241	.145799
.176	.093074	.209	.119083	.242	.146655
.177	.093836	.210	.119897	.243	.147512
.178	.094601	.211	.120712	.244	.148371
.179	.095366	.212	.121529	.245	.149230
.180	.096134	.213	.122347	.246	.150091
.181	.096903	.214	.123167	.247	.150953
.182	.097674	.215	.123988	.248	.151816
.183	.098447	.216	.124810	.249	.152680
.184	.099221	.217	.125634	.250	.153546
.185	.099997	.218	.126459	.251	.154412
.186	.100774	.219	.127285	.252	.155280
.187	.101553	.220	.128113	.253	.156149
.188	.102334	.221	.128942	.254	.157019
.189	.103116	.222	.129773	.255	.157890
.190	.103900	.223	.130605	.256	.158762
.191	.104685	.224	.131438	.257	.159636
.192	.105472	.225	.132272	.258	.160510
.193	.106261	.226	.133108	.259	.161386
.194	.107051	.227	.133945	.260	.162263
.195	.107842	.228	.134784	.261	.163140
.196	.108636	.229	.135624	.262	.164019
.197	.109430	.230	.136465	.263	.164899
.198	.110226	.231	.137307	.264	.165780
.199	.111024	.232	.138150	.265	.166663
.200	.111823	.233	.138995	.266	.167546
.201	.112624	.234	.139841	.267	.168430

The Areas of the Segments of a Circle.

Verſed Sine.	*Seg. Area*	Ver-ſed Sine.	*Seg. Area.*	Ver-ſed Sine.	*Seg. Area.*
.268	.169315	.301	.199085	.334	.229801
.269	.170202	.302	.200003	.335	.230745
.270	.171089	.303	.200922	.336	.231689
.271	.171978	.304	.201841	.337	.232634
.272	.172867	.305	.202761	.338	.233580
.273	.173758	.306	.203683	.339	.234526
.274	.174649	.307	.204605	.340	.235473
.275	.175542	.308	.205527	.341	.236421
.276	.176435	.309	.206451	.342	.237369
.277	.177330	.310	.207376	.343	.238318
.278	.178225	.311	.208301	.344	.239268
.279	.179122	.312	.209227	.345	.240218
.280	.180019	.313	.210154	.346	.241169
.281	.180918	.314	.211082	.347	.242121
.282	.181817	.315	.212011	.348	.243074
.283	.182718	.316	.212940	.349	.244026
.284	.183619	.317	.213871	.350	.244980
.285	.184521	.318	.214802	.351	.245934
.286	.185425	.319	.215733	.352	.246889
.287	.186329	.320	.216666	.353	.247845
.288	.187234	.321	.217599	.354	.248801
.289	.188140	.322	.218533	.355	.249757
.290	.189047	.323	.219468	.356	.250715
.291	.189955	.324	.220404	.357	.251673
.292	.190864	.325	.221340	.358	.252631
.293	.191775	.326	.222277	.359	.253590
.294	.192684	.327	.223215	.360	.254550
.295	.193596	.328	.224154	.361	.255510
.296	.194509	.329	.225093	.362	.256471
.297	.195422	.330	.226033	.363	.257433
.298	.196337	.331	.226974	.364	.258395
.299	.197252	.332	.227915	.365	.259357
.300	.198168	.333	.228858	.366	.260320

The Areas of the Segments of a Circle.

Verſed Sine.	*Seg. Area.*	Verſed Sine.	*Seg. Area.*	Verſed Sine.	*Seg. Area.*
.367	.261284	.400	.293369	.433	.325900
.368	.262248	.401	.294349	.434	.326892
.369	.263213	.402	.295330	.435	.327882
.370	.264178	.403	.296311	.436	.328874
.371	.265144	.404	.297292	.437	.329866
.372	.266111	.405	.298273	.438	.330858
.373	.267078	.406	.299255	.439	.331850
.374	.268045	.407	.300238	.440	.332843
.375	.269013	.408	.301220	.441	.333836
.376	.269982	.409	.302203	.442	.334829
.377	.270951	.410	.303187	.443	.335822
.378	.271920	.411	.304171	.444	.336816
.379	.272890	.412	.305155	.445	.337810
.380	.273861	.413	.306140	.446	.338804
.381	.274832	.414	.307125	.447	.339798
.382	.275803	.415	.308110	.448	.340793
.383	.276775	.416	.309095	.449	.341787
.384	.277748	.417	.310081	.450	.342782
.385	.278721	.418	.311068	.451	.343777
.386	.279694	.419	.312054	.452	.344772
.387	.280668	.420	.313041	.453	.345768
.388	.281642	.421	.314029	.454	.346764
.389	.282617	.422	.315016	.455	.347759
.390	.283592	.423	.316004	.456	.348755
.391	.284568	.424	.316992	.457	.349752
.392	.285544	.425	.317981	.458	.350748
.393	.286521	.426	.318970	.459	.351745
.394	.287498	.427	.319959	.46	.352741
.395	.288476	.428	.320948	.461	.353739
.396	.289453	.429	.321938	.462	.354736
.397	.290432	.430	.322928	.463	.355732
.398	.291411	.431	.323918	.464	.356730
.399	.292390	.432	.324909	.465	.357727

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The Areas of the Segments of a Circle.

Verſed Sine.	*Seg. Area.*	Verſed Sine.	*Seg. Area.*	Verſed Sine.	*Seg. Area.*
.466	.358725	.478	.370706	.490	.382699
.467	.359723	.479	.371705	.491	.383699
.468	.360721	.480	.372764	.492	.384699
.469	.361719	.481	.373703	.493	.385699
.470	.362717	.482	.374702	.494	.386699
.471	.363715	.483	.375702	.495	.387699
.472	.364713	.484	.376702	.496	.388699
.473	.365712	.485	.377701	.497	.389699
.474	.366710	.486	.378701	.498	.390699
.475	.367709	.487	.379700	.499	.391699
.476	.368708	.488	.380700	.500	.392699
.477	.369707	.489	.381699		

THE END.

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